DETERMINANTAL POINT PROCESSES AND FERMIONS ON COMPLEX MANIFOLDS: BULK UNIVERSALITY

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ABSTRACT. Determinantal point processes on a compact complex manifold X are considered in the limit of many particles. The correlation kernels of the processes are the Bergman kernels associated to a a high power of a given Hermitian holomorphic line bundle L over X. It is shown that the defining random measure on X of the process, describing the particle locations, converges in probability towards a pluripotential equilibrium measure, expressed as a Monge-Ampere measure. Its smooth fluctuations in the bulk are shown to be asymptotically normal and the limiting variance is explicitly computed. A scaling limit of the correlation functions is shown to be universal and expressed in terms of (the higher dimensional analog of) the Ginibre ensemble. This setting applies in particular to normal random matrix ensembles and multivariate orthogonal polynomials. Relation to phase transitions, direct image bundles and tunneling of ground state fermions in strong magnetic fields (i.e. exponentially small eigenvalues of the Dolbeault Laplacian) are also explored.

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1. INTRODUCTION

The systematic study of *determinantal point processes* was initiated by Macchi [45] in the seventies who called them *fermionic* point processes, inspired by the properties of fermion gases in statistical (quantum) mechanics. For general reviews see [60, 38, 40]. The theory concerns ensembles of "particle configurations" on a given space X which exhibit *repulsion.* An important class of such processes are the determinantal *projectional* processes, which may be defined by a probability measure

on the $N-$fold product X^N, the "configuration space of N particles on X", with the property that its density may be written as

$$\mathcal{P}(x_1, ..., x_N) = \frac{1}{N!} \det(\widetilde{K}(x_i, x_j)), \tag{1.1}$$

where the kernel $\widetilde{K}$ is the integral kernel of an orthogonal projection operator onto a vector space of dimension N. As a consequence the probability distributions vanish for a configuration $(x_1, ..., x_N)$ of points x_i as soon as two points coincide, explaining the repulsive behavior of the ensemble. As it turns out in many situations such ensembles are *critical* in the sense that they naturally appear in sequences with N, the number of particles, tending to infinity in such a way that a well-defined limiting ensemble may be extracted. Moreover, large classes of such sequences of ensembles often give rise to one and the same limit. This is the phenomena of *universality* (see [21] for a nice survey). Perhaps its most famous illustration is given by ensembles of $N \times N$ *Hermitian random matrices* whose eigenvalues, in the large N limit, determine a unique determinantal point process on the real line. This latter process has also been conjectured to describe the statistics of the zeroes of the Riemann zeta function ref, as well as statistics of quantum systems whose classical dynamics is chaotic(references and more recent relations to random growth and tiling problems may be found in [40]).

The present paper concerns a general class of such critical ensembles, where the space X is a compact complex manifold equipped with an holomorphic line bundle L with a given Hermitian metric locally represented as $e^{-\phi}$, where ϕ is called a "weight" on L. The kernel $\widetilde{K}$ defining the ensemble may then be identified with the orthogonal projection onto the space of global holomorphic sections $H^0(X, L)$ of L (with respect to a local unitary frame of $(L, e^{-\phi})$). In this setting the limit of a large number of particles corresponds to the limit when the line bundle L is replaced by a large tensor power. When X is the complex projective space this setting is just a geometric formulation of the theory of (weighted) multivariate orthogonal polynomials (see section 2). In mathematical physics terminology $H^0(X, L)$ may be identified with the quantum ground state space of a single fermion (complex spinor) on X subject to an exterior magnetic field and the density 1.1 is the squared probability amplitude for the corresponding maximally filled many particle state.

Already in the simplest case when X is the complex projective line (viewed as the one-point compactification of $\mathbb{C}$) the corresponding ensemble is remarkably rich and admits at least three different well-known descriptions in terms of *(1) normal random matrices, (2) a free fermion gas, (3) a Coulomb gas* of repelling electric charges. See section 2 for a quick review of this fact.

While there are quite recent result concerning this special case, both in mathematics and physics, there seems to be almost no previous results in the higher dimensional situation studied in the present paper (and very

few results in the general one-dimensional case where X is a compact Riemann surface). For one reference see the recent paper [53]. As it turns out, the main new feature that appears in higher dimensions is that the role of the Laplace operator in one complex dimension (which expresses the limiting expected density of particles) is played by the *fully non-linear Monge-Ampere operator,* which is the subject of (complex) *pluripotential* theory [42, 33, 34]. In fact, one of the motivations for the present paper is to develop a Coulomb gas type descriptions of a gas of free fermions on complex manifolds and conversely to provide a statistical mechanical interpretation of complex pluripotential theory.

Yet another motivation comes from approximation theory where configurations $(x_1, ...x_N)$ appear as *interpolation nodes* on X and a configuration maximizing a functional of the form 1.1 is known to have optimal interpolation properties in a certain sense [36, 56]. Sequences of configurations, with N tending to infinity, then appear naturally in discretization schemes. Moreover, as shown very recently in [13] any such optimal sequence equidistributes asymptotically on the corresponding equilibrium measure. This fact should be compared with Theorem 1.4 in the present paper which shows that, with high probability, the same equidistribution property holds for *random* configurations of the corresponding ensemble.

One final motivation comes from the study by Shiffman, Zelditch and coworkers of random zeroes of holomorphic sections of positive line bundles, where many statistical results have been obtained and where a key role is played by Bergman kernels (cf. [17, 57, 58]).

The main results of the present paper, to which we now turn, concern the universality of the correlation functions and fluctuations in the "bulk". The relation to phase transitions (with respect to perturbations of the weight ϕ) is also explored.

1.1. **Statement of the main results.** Let L be a holomorphic line bundle over a compact complex manifold X of dimension n. Denote by $H^0(X, L)$ the vector space of all global holomorphic sections on X with values in L and write $N := \dim H^0(X, L)$. Fixing an Hermitian metric on L, represented by a *weight* ϕ, and a suitable measure μ induces an inner product on $H^0(X, L)$ defined by

$$\|s\|_\phi^2 := \int_X |s|^2 e^{-\phi}\mu$$

(we are abusing notation slightly; see section 1.4). We will denote the corresponding Hilbert space by $\mathcal{H}(X, L)$ and its *Bergman kernel* by K, which is the integral kernel of the orthogonal projection $\mathcal{C}^\infty(X, L) \to H^0(X, L)$:

$$K(x, y) = \sum_i s_i(x) \otimes \overline{s_i(y)}, \tag{1.2}$$

where (s_i) is an orthonormal bases in $\mathcal{H}(X, L)$.

As is well-known this setup induces a probability measure γ_P on the $N-$fold product X^N whose density (w.r.t. μ^N) is defined as the determinant of an $N \times N$ matrix:

$$\mathcal{P}(x_1, ..., x_N) := \frac{1}{N!}\det(K(x_i, x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))}) \tag{1.3}$$

The main object of study in the present paper is the large k asymptotics of the probability space (X^N, γ_P), when L is replaced by its kth tensor power (written as kL in additive notation) equipped with the induced weight $k\phi$. In the following a subindex k will be used to indicate the the dependence on the parameter k. We will always assume that L is *big,* i.e that

$$N_k := \dim H^0(X, kL) = Ck^n + o(k^{n-1}), \; C > 0$$

(where the constant C is usually called the *volume* of L). Moreover, the main results concern weighted measured (ϕ, μ) which are *strongly regular.* This will mean that the weight ϕ is $\mathcal{C}^{1,1}$-smooth, i.e. it is differentiable and all of its first partial derivatives are locally Lipschitz continuous, and the measure $\mu = \omega_n$ is the volume form of a continuous metric ω on X. When X is the complex projective space $X := \mathbb{P}^n$ and L the hyperplane line bundle $\mathcal{O}(1)$ (so that $H^0(X, kL)$ may be identified with the space of all polynomials of total degree at most k in $\mathbb{C}^n$) we also allow ω_n to be the Lebesgue measure on the affine piece $\mathbb{C}^n$ as long as ϕ has super logarithmic growth (formula 2.4). In the sequel we will usually assume that (ϕ, μ) is strongly regular. However, occasionally we will explicitly point out that a result is valid for a *weakly* regular pair (ϕ, μ), where for example μ is allowed to be supported on a totally real domain (see section 1.4). As a guide line, results concerning the "macroscopic regime" will be shown to hold in the weakly regular situation, while the results in the "microscopic regime", concerning lenghth scales of the order $k^{-1/2}$, only hold in the strongly regular case.

1.1.1. *Correlation functions and the equilibrium measure.* As is well known all the *$m-$point correlation functions* $\rho_k^{(m)}$, where $1 \leq m \leq N_k$, of the ensemble above may be expressed as (weighted) determinants of $K_k(x_i, x_j)$. In particular,

$$\rho_k^{(1)}(x) = K_k(x, x)e^{-k\phi(x)}, \quad \rho^{(2).c}(x, y) = -\left|K_k(x, y)\right|^2 e^{-k\phi(x)}e^{-k\phi(y)},$$

where $\rho^{(2).c}$ is the *connected* 2-point correlation function (see section 6.1). As recently shown in [9], in the strongly regular case,

$$k^{-n}\rho_k^{(1)}\omega_n \to \mu_{\phi_e}, \tag{1.4}$$

weakly, when $k \to \infty$, where μ_{ϕ_e} is the pluripotential *equilibrium measure* (of (X, ϕ)), which may be written as the Monge-Ampere measure $(dd^c\phi_e)^n/n!$ of the equilibrium weight ϕ_e. In fact, as later shown in [13] the convergence holds for *weakly* regular weighted measures (ϕ, μ). However, in the strongly regular setting *point-wise* convergence actually holds

in the sense that there is a subset of X that will be called the *bulk* (of (X, ϕ)) such that

$$k^{-n}\rho_k^{(1)}(x) \to \det_\omega(dd^c\phi)(x), \quad x \text{ in the bulk}$$

and converges to zero almost everywhere in the complement of the bulk. The precise definition of the bulk is given in section 3 - for now we will just mention that $dd^c\phi > 0$ in the bulk, but unless $dd^c\phi \geq 0$ globally on X the bulk is usually strictly contained in $X(0) := \{dd^c\phi > 0\}$.

The following theorem gives the scaling asymptotics, around a fixed point x in the bulk, of the Bergman kernel. It is expressed in terms of "normal" local coordinates z centered at x and a "normal" trivialization of L, i.e such that

$$\omega(z) = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge \overline{dz_i} + ..., \quad \phi(z) = \sum_{i=1}^{n} \lambda_i \left|z_i\right|^2 + ... \tag{1.5}$$

where the dots indicate "higher order terms". Hence, λ_i are the eigenvalues of the curvature form $dd^c\phi$ w.r.t the metric ω and we denote the corresponding diagonal matrix by λ.

Theorem 1.1. *Assume that (ϕ, ω_n) is a strongly regular weighted measure. Let x be a fixed point in the bulk and take "normal" local coordinates z centered at x and a "normal" trivialization of L as above. Then*

$$k^{-n}K_k(k^{-1/2}z, k^{-1/2}w) \to \frac{\det \lambda}{\pi^n} e^{\langle \lambda z, w\rangle}$$

in the $\mathcal{C}^\infty-$topology on compact subsets of $\mathbb{C}_z^n \times \mathbb{C}_w^n$. In particular, the connected 2-point function has the following scaling asymptotics

$$-k^{-2n}\rho_k^{(2).c}(k^{-1/2}z, k^{-1/2}w) \to (\frac{\det \lambda}{\pi^n})^2 e^{-\sum_{i=1}^n \lambda_i |z_i - w_i|^2}$$

uniformly on compacts of $\mathbb{C}_z^n \times \mathbb{C}_w^n$.

The previous theorem may on one hand be interpreted as a "localization" result, in the sense that the limit is expressed in terms of local data (the curvature of $dd^c\phi$ at the fixed point). On the other hand, it can be seen as a "universality" result (see [21] for a general discussion of universality in mathematics and physics). Indeed, scaling the coordinates further in order to make the metric $dd^c\phi$ at the fixed point the "yard stick" the limiting kernel becomes independent of the ensemble (and coincides with the Bergman kernel of Fock space). When $n = 1$ the corresponding limiting one-dimensional determinantal point process was studied by Ginibre, who showed that it appears from a scaling limit of random complex matrices with independent complex Gaussian entries.

As a corollary the following analog of a well-known universality result for the Hermitian random matrix model (where the limiting kernel is the sine kernel) is obtained:

Corollary 1.2. *Let ϕ be a function in $\mathcal{C}^{1,1}(\mathbb{C})$ with super logarithmic growth and denote by $\rho_k^{(\cdot)\cdot}$ the eigenvalue correlation functions of the associated normal random matrix model (see section 2.2.1). Then the following universal relation holds when the rank $N = k+1$ of the matrices tends to infinity:*

$$-\frac{\rho_k^{(2).c}(z_0 + \frac{z}{\sqrt{\rho_k^{(1)}(z_0)}}, z_0 + \frac{w}{\sqrt{\rho_k^{(1)}(z_0)}},)}{\left(\rho_k^{(1)}(z_0)\right)^2} \to e^{-|z-w|^2}$$

uniformly on compacts of $\mathbb{C}\times\mathbb{C}$, when z_0 is a fixed point in the eigenvalue plane $\mathbb{C}$.

The next theorem implies that the correlations are short range on *macroscopic* length scales in the bulk.

Theorem 1.3. *Assume that (ϕ, ω_n) is a strongly regular weighted measure. Let E be a compact subset of the interior of the bulk. Then there is a constant C (depending on E) such that the following estimate holds for all pairs (x, y) such that either x or y is in E :*

$$-k^{-2n}\rho_k^{(2).c}(x, y) \leq Ce^{-\sqrt{k}d(x,y)/C}$$

for all k, where $d(x, y)$ is the distance function with respect to a fixed smooth metric on X.

1.1.2. *Fluctuations of linear statistics.* Consider the random measure (i.e. a measure valued random variable) defined by

$$(x_1, ..., x_N) \mapsto \sum_{i=1}^{N} \delta_{x_i}, \tag{1.6}$$

Its expected value is the one point correlation measure $\rho^{(1)}\omega_n$. To get a real-valued random variable one fixes a function u on X and defines the random variable $\mathcal{N}[u]$ by contraction:

$$\mathcal{N}[u](x_1, ..., x_N) := u(x_1) + ... + u(x_N),$$

often called a *linear statistic* in the literature. In particular, if $u = 1_E$ is the characteristic function of a subset E of X, then $\mathcal{N}[u](x_1, ..., x_N)$ counts the number of x_i contained in E. By 1.4 the expected value of the normalized random measure 1.6 converges weakly to the *equilibrium measure* of (X, ϕ). The following theorem says that one actually has convergence in *probability:*

Theorem 1.4. *Assume that (ϕ, μ) is a weakly regular weighted measure. Let u be a continuous function on (X, μ). Then*

$$k^{-n}\mathcal{N}_k[u] \to \int_X \mu_\phi u \tag{1.7}$$

in probability when k tends to infinity at a rate of order $O(k^{-n})$, i.e.

$$Prob_k(\{(x_1,...,x_{N_k}):\left|k^{-n}(u(x_1)+....+u(x_{N_k}))-\int_X \mu_\phi u\right|>\epsilon\})\leq\frac{C}{\epsilon^2k^n}$$

for some constant C independent of ϵ and k.

Note that it follows from basic integration theory that the convergence also holds if u is the characteristic function of a, say smooth, domain E in X, as long as the limiting measure μ_ϕ is absolutely continuous (w.r.t. a smooth volume form). In particular, this happens in the strongly regular case. To get convergence in probability, i.e. to see that a (weak) "law of large numbers" holds, the following simple variance estimate is used:

$$\mathrm{Var}(\mathcal{N}_k[u]):=\mathbb{E}(\widetilde{\mathcal{N}_k}[u])^2)=O(k^n)$$

for any u as above, where $\widetilde{\mathcal{N}_k}[u]$ is the "fluctuation"

$$\widetilde{\mathcal{N}_k}[u]:=\mathcal{N}_k[u]-\mathbb{E}(\mathcal{N}_k[u])$$

of the random variable $\mathcal{N}_k[u]$.

Next, the fluctuations in the bulk are considered for functions u which are suitable smooth. The main result is the following central limit theorem, formulated in terms of the Laplace transform of the law on $\mathbb{R}$ of $\widetilde{\mathcal{N}_k}[u]$:

Theorem 1.5. *Assume that (ϕ,ω_n) is a strongly regular weighted measure. Let u be a $\mathcal{C}^{1,1}-$ smooth function on X supported in the bulk. Then the sequence $\mathbb{E}(e^{-tk^{-(n-1)/2}\widetilde{\mathcal{N}_k}[u],})$ converges to a Gaussian function in t. More precisely,*

$$\lim_{k\to\infty}\log\mathbb{E}(e^{-tk^{-(n-1)/2}\widetilde{\mathcal{N}_k}[u],})=(\|du\|^2_{dd^c\phi})t^2/2$$

in the $\mathcal{C}^2-$topology on any given compact subset in $\mathbb{R}_t$. If u is merely $\mathcal{C}^1-$ smooth then the point-wise convergence of the second derivatives at $t=0$ still holds.

Corollary 1.6. *Let u be a $\mathcal{C}^1-$smooth function on X supported in the bulk. Then (i) the variance of the random variable $\mathcal{N}[u]$ has the following asymptotics*

$$Var_k(\mathcal{N}[u])=(\|du\|^2_{dd^c\phi})k^{n-1}+o(k^{n-1}).$$

(ii) Assume further that u is $\mathcal{C}^{1,1}-$smooth. Then the normalized random variable $\widetilde{\mathcal{N}_k}[u]/\sqrt{Var(\mathcal{N}_k[u])}$ converges in distribution to the standard normal variable with mean zero and unit variance i.e.

$$Prob_k\{\frac{(u(x_1)-\mathbb{E}u(x_1))+....+(u(x_{N_k})-\mathbb{E}u(x_{N_k})}{\sqrt{Var(u(x_1)+...+u(x_{N_k}))}}\}\leq s)\to\frac{1}{\sqrt{\pi}}\int_{-\infty}^s e^{-x^2}dx$$

Just like Theorem 1.1 the previous results may be interpreted as universality results (compare the discussion in [21]). Note in particular that the one-dimension case $(n=1)$, i.e. when X is a Riemann surface, is singled out by the fact that the variance is independent of the weight ϕ.

In general dimensions, the previous result essentially says that the fluctuations converge in distribution to the *Gaussian free field* [59] in the bulk (w.r.t. the metric $dd^c\phi$). Compare [50, 2] for precise statements in one-dimensional situations.

On the other hand, as there may, as pointed out below, appear second order phase transitions (when the weight ϕ is perturbed), a central limit theorem for general smooth functions u, is not to be expected.

1.1.3. *Free energy and phase transitions.* Next, we will interpret some of the previous results in terms of *phase transitions.* In statistical mechanical terms the probability space on X^N may be realized as a *Boltzmann-Gibbs ensemble* in the following way. The line bundle L induces an "internal energy" on the "configuration space" X^N of N identical particles distributed on the complex manifold X :

$$E_{int}((x_1, ..., x_N) := -\log(|\det S(x_1, .x_N)|^2),$$

where $\det S(x_1, .x_N)$ is the section of the pulled-back line bundle $L^{\boxtimes N}$ over X^N defined as

$$\det(S)(x_1, .x_N) := \det_{1\leq i,j\leq N}(s_i(x_i))_{i,j} \tag{1.8}$$

in terms of a fixed base $S = (s_i)$ in $H^0(X, L)$. This means that E_{int} is a weight for the dual of $L^{\boxtimes N}$, which is canonically defined by the line bundle L up to an additive constant. The weight ϕ, in turn, plays the role of an exterior potential and hence induces the following *"external* energy"

$$E_{ext}(x_1, ..., x_N) := \phi(x_1) + ... + \phi(x_N),$$

which is a weight for $L^{\boxtimes N}$. Summing gives the total energy $E_\phi = E_{int} + E_{ext}$ which is a hence a *function* on the configuration space X^N. This energy function (=Hamiltonian) determines, in the usual way, the probability density of the corresponding *Boltzmann-Gibbs ensemble* (which coincides with 1.3):

$$\mathcal{P}(x_1, ..., x_N) := \frac{e^{-E_\phi(x_1,...,x_N)}}{\mathcal{Z}[\phi]}, \tag{1.9}$$

where the normalization factor $\mathcal{Z}[\phi]$ is called the *partition function* and $\mathcal{F}[\phi]= -\log \mathcal{Z}[\phi]$ is called the *free energy.*

Given a positive integer k one now obtains a sequence of *Boltzmann-Gibbs* ensembles as above. The corresponding large k limit may then be interpreted the limit when the number of particles N_k tends to infinity, i.e. as a "thermodynamical" limit. In particular, the triple (L, ϕ, ω_n) induces the following sequence of functionals on the space of all weights for L :

$$\phi \mapsto \mathcal{F}[k\phi]$$

which are canonical up to additive constants. Hence, their functional derivatives (differentials) may be represented by canonical measures on

X which turns out to coincide with the expectation of the 1−point correlation measure. To fix the value $\mathcal{F}[k\phi]$ we take the base (s_i) above to be orthonormal with respect to a weighted measure $(k\phi_0, \omega_n)$. The main result in [12, 13] may now be translated to:

Theorem. *Let (ϕ, μ) be a weakly regular weighted measure and let ϕ_0 be a fixed "reference weight" and consider the corresponding sequence of free energies $\mathcal{F}[k\phi]$ (relative $k\phi_0$). Then there is a functional $\mathcal{F}_\infty$ such that*

$$k^{-(n+1)}\mathcal{F}[k\phi] \to \mathcal{F}_\infty[\phi].$$

Moreover, the weak convergence at the level of differentials also holds:

$$\frac{d}{dt}(k^{-(n+1)}\mathcal{F}[k(\phi+tu)]) \to \frac{d}{dt}\mathcal{F}_\infty[\phi+tu] = \int_X \mu_\phi u$$

for any continuous function u on X.

For a formula of the functional $\mathcal{F}_\infty[\phi]$ see section 3.1. The previous result should be seen in the light of the theory of *phase transitions.* These may be defined in terms of regularity properties of the free energy when the system is perturbed. More precisely, according to the Ehrenfest classification, a family of Gibbs ensembles, depending on a parameter t, is said to have a phase transition of order m at $t = 0$ if the m th derivative av the corresponding free energy $\mathcal{F}_t$ is the lowest derivative to not exist at $t = 0$. In the present case one consider the family above obtained by varying the weight: $\phi_t = \phi + tu$ (corresponding to linear perturbations of the exterior potential). From this point of view the previous theorem implies the absence of phase transitions of the first order. However, as observed in [12] it may happen that the second derivatives of $\mathcal{F}_\infty[\phi+tu]$ does *not* exist, i.e. that there is a phase transitions of the *second* order.

Still, the following result implies that if u is assumed to be supported in the pseudo-interior of the bulk of (X, ϕ), then this does not happen.

Theorem 1.7. *Fix a continuous volume form ω_n and $\mathcal{C}^{1,1}$-family of weights ϕ_t so that (ϕ_t, ω_n) is a family of weighted measures, where the $t-$derivatives of ϕ_t are supported in the bulk of (X, ϕ_0). Then*

$$k^{-(n-1)}\frac{\partial^2\mathcal{F}_k}{\partial t\partial\bar{t}} \to \frac{\partial^2\mathcal{F}_\infty(t)}{\partial t\partial\bar{t}}$$

when k tends to infinity.

Moreover, the limit above is explicitly given using

$$d_t d_t^c\mathcal{F}_\infty[\phi_t] = t_*(dd^c\Phi)^{n+1}/(n+1)!,$$

where, locally, $\Phi(z,t) = \phi_t(z)$ is assumed $\mathcal{C}^{1,1}$-smooth. The relation to direct image bundles is briefly explored in section 7.3.

As shown in [12] $\mathcal{F}_\infty[\phi+tu]$ is $\mathcal{C}^{1,1}$-smooth w.r.t. any, say smooth, function u. This motivates the following

Conjecture 1.8. *Let (ϕ,μ) be a weakly regular weighted measure and u a Lipschitz continuous function on X. Then there is a constant C (independent of k) such that*

$$0 \leq -\frac{d^2\mathcal{F}_k[k(\phi+u]}{d^2t}_{t=0} \leq Ck^{n-1}$$

As the left hand side above is precisely the variance $\mathrm{Var}_k(\mathcal{N}[u])$ the conjecture amounts to having a general variance estimate improving those above. The proof of the following special case of the conjecture will appear elsewhere:

Theorem 1.9. *Let X be a complex subvariety of a complex projective space let and L be the restriction to X of the corresponding hyper plane line bundle. Let (ϕ,μ) be a weakly regular weighted measure such that μ is supported on the corresponding real projective variety $X(\mathbb{R})$. Then the previous conjecture holds.*

The previous theorem in particular applies to the classical setting of multivariate orthogonal polynomials on domains in $\mathbb{R}^n$ if one takes X as the $n-$dimensional complex projective space (see section 2).

Finally, it should be pointed out that in case ϕ is smooth (i.e. in $\mathcal{C}^\infty$) and the measure $\mu = \omega_n$ is a smooth volume form on X it was shown in [9] that the Bergman kernel $K(x,y)$ admits a local complete asymptotic expansion when x and y are in the interior of the bulk. As a consequence, many of the results above may be refined in this case. Similarly, when ϕ is smooth most of the present results may be generalized to the case when $\mu = 1_E\omega_n$ where E is a domain in X with smooth boundary. But is then important to take the definition of the bulk to be the interiour of the set $E \cap D \cap X(0)$. In this context, it would be very interesting to understand the scaling asymptotics of the correlations at the *boundary* of E, but this seems to require new ideas. For a special case where this has been carried out see [7].

Remark 1.10. The previous results are actually shown to hold in a more general setting where $(kL, k\phi)$ is replaced by $(kL+F, k\phi+\phi_F)$ were (F,ϕ_F) is a Hermitian holomorphic line bundle with suitable regularity properties. In fact, this flexibility will allow us to pass directly from variance asymptotics to a central limit theorem.

1.2. **Tunneling of fermions and analytic torsion.** Finally, we will announce a result that relates the exponentially small eigenvalues of the twisted Dolbeault Laplacian (corresponding to the Pauli operator coupled to a magnetic field in the mathematical physics literature) to the free energy functional above. The proof will appear elsewhere.

In order to state the result first recall that the $\overline{\partial}-$operator gives rise to the *Dolbeault complex* $(\Omega^{0,\cdot}(X,L), \overline{\partial})$:

(1.10)

$$... \rightarrow \; \Omega^{0,q-1}(X,L) \; \overset{\overline{\partial}}{\rightarrow} \; \Omega^{0,q}(X,L) \; \overset{\overline{\partial}}{\rightarrow} \; \Omega^{0,q+1}(X,L) \; \rightarrow ...(\overline{\partial}^2 = 0)$$

where $\Omega^{0,q}(X,L)$ denotes the space of all smooth $(0,q)-$ forms with values in L. Fixing a smooth metric ω and a smooth weight ϕ on L induces inner products on the previous complex. The corresponding formal adjoint $\overline{\partial}^*$ then defines a dual complex (by reversing the arrows). The *Dolbeault Laplacian* Δ on $\Omega^{0,\cdot}(X,L)$may now be defined as the square of the corresponding *Dirac operator* $\overline{\partial}+\overline{\partial}^*$. Hence, it decomposes according to degree as

$$\Delta=\overline{\partial}^*\overline{\partial}+\overline{\partial}^*\overline{\partial}=\bigoplus_{q=1}^{n}\Delta^{(q)}$$

Finally, replacing (L,ϕ) be $(kL,k\phi)$ gives rise to a sequence Δ_k of Dolbeault Laplacians.

Theorem 1.11. *Let (L,ϕ) be an ample line bundle with a smooth weight ϕ over an Hermitian manifold (X,ω). Then for any $p>0$*

$$k^{-(n+1)}\sum_{q=1}^{n}q(-1)^q\log\det(1_{]0,k^{1-p}[}(\Delta_k^{(q)}))\rightarrow-\mathcal{E}[\phi_e,\phi], \tag{1.11}$$

where $\mathcal{E}$ is the bi-functional 3.8. Moreover,

$$\log\det(1_{]0,k^{1-p}[}(\Delta_k^{(q)})+o(k^{(n+1)})=\log\widetilde{\det}(\Delta_k^{(q)}) \tag{1.12}$$

where the right hand side is the zeta function regularized determinant of the restriction of $(\Delta_k^{(q)})$ to the orthogonal complement of its null-space.

The expression $\mathcal{E}[\psi,\psi']$ appearing above may be defined as a *Bott-Chern class.* It is related to the free energy functional above by the relation

$$\mathcal{F}_\infty[\phi]=\mathcal{E}[\phi_e,\phi_0]$$

if the reference weight ϕ_0 is taken to be psh. Recall also that L is ample precisely when it admits a smooth weight ϕ with positive curvature. By the Hörmander-Kodaira estimates there is a uniform positive lower bound on the first positive eigenvalue of $\Delta_k^{(q)}$ for any q for such a weight. This is consistent with the fact that $\mathcal{E}[\phi_e,\phi]=0$ when ϕ has positive curvature. The last statement of the previous theorem is a fairly straight-forward generalization of [16]. Together 1.11 and 1.12 say that the *Ray-Singer analytic torsion* of the Dolbeault complex associated to a high power kL also tends to $-\mathcal{E}[\phi_e,\phi]$ when divided by $k^{(n+1)}$. This latter fact was first proved in [12] under the further assumption that ω be a Kähler metric, which allowed the use of exact results (i.e. for k fixed) in [15] for the Quillen metric.

In the one-dimensional case 1.11 is equivalent to

$$-k^{-2}\sum_i\log\lambda_{i,k}\rightarrow\left\|d(\phi-\phi_e)\right\|^2/2,$$

where the sum is taken over all positive eigenvalues of the Dolbeault Laplacian acting on smooth sections with values in kL, that are less than k^{-p} for any fixed positive number p.

1.3. **Relation to previous results.** The main point of the present paper is to apply techniques from complex geometry/analysis, notable $\overline{\partial}$-estimates, to determinantal point processes. It should be emphasized that in the case of a smooth weight ϕ corresponding to a *smooth positively curved* metric on L the results on the corresponding Bergman kernels are well-known and go back to the work of Tian, Zelditch, Catlin and others. For the decay estimate in Theorem 1.3 in a $\mathbb{C}^n$-setting see [24, 44]. Note that by an example of M.Christ the rate of decay in Theorem 1.3 is essentially optimal. The extension to smooth, but not unnecessarily positively curved metrics and the relation to equilibrium measures was initiated recently in [9, 8] and then developed to less regular weights and measures in [12, 13].

The bulk universality in corollary 1.2 should be compared with the case when the reference measure is the usual invariant measure on $\mathbb{R}$. The corresponding bulk universality, at length scales of the order k^{-1}, is then well-known in the context of random Hermitian matrices and the corresponding limiting kernel is then the sine kernel. In this context the bulk is usually defined as the maximal open set in $\mathbb{R}$ where the corresponding equilibrium measure has a positive continuous density (cf. [49] where mean-field theory methods are used) and [23] for the real-analytic case, where Riemann-Hilbert methods are used. For the convergence in probability in this "real" case (which is a special case of Theorem 1.4) see [47] and references therein. The analog in the real case of Corollary 1.6, concerning fluctuations for a convex weight on $\mathbb{R}$ (but for "general" u) was obtained in [39]. See also [48] where it is shown that the corresponding central limit theorem for "general" u on may not hold for non-convex weights in $\mathbb{R}$. This fact should be compared with the fact pointed out in the present paper that there are perturbation tu which give rise to second order phase transitions. Note also that in the case of the translational invariant reference measure on S^1 *third* order phase transitions has been appeared in the physics litterature in the context of lattice gauge theory [32], as well as in combinatorics [41]. During the preparation of the present paper the preprint [1] appeared where Corollary 1.6 was obtained for real-analytic ϕ in $\mathbb{C}$, also by refining the analysis in [8], but combined with a more combinatorial argument using expansion of cumulants. For the special case where $\phi = |z|^2$, but "general" u are considered, see [50]. For a general central limit theorem of the form (ii) in Corollary 1.6 for general determinantal point process (given estimates on the variances) see [61]. The variance asymptotics in (i) in Corollary 1.6 is also closely related to the results in [14] on direct image bundles (compare section 7.3).

Finally, it should be emphasized that in the smooth positively curved case (with a smooth reference measure) Bergman kernel asymptotics have already been applied and developed extensively by Shiffman-Zelditch and their collaborators in the context of random zeroes of holomorphic sections (defined with respect to the Gaussian probability measure on the

Hilbert space $\mathcal{H}(X, kL)$). For example, universality of the corresponding correlation functions was proved in [17] and a central limit theorem (when $n = 1$) was obtained in [58].

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1.4. **Notation and general setup.**

Weights on line bundles[1]. Let L be a holomorphic line bundle over a compact complex manifold X. We will represent an Hermitian metric on L by its *weight* ϕ. In practice, ϕ may be defined as certain collection of *local* functions. Namely, let s^U be a local holomorphic trivializing section of L over an open set U (i.e. $s^U(x) \neq 0$ for x in U). Then locally, $\left|s^U(z)\right|_\phi^2 =: e^{-\phi^U(z)}$. If α is a holomorphic *section* with values in L, then over U it may be locally written as $\alpha = f^U \cdot s^U$, where f^U is a local holomorphic *function.* In order to simplify the notation we will usually omit the dependence on the set U and s^U and simply say that f is a local holomorphic function representing the section α.

The point-wise norm of α may then be locally expressed as

$$|\alpha|_\phi^2 = |f|^2 e^{-\phi}, \tag{1.13}$$

but it should be emphasized that it defines a *global* function on X.

The canonical curvature two-form of L is the global form on X, locally expressed as $\partial\overline{\partial}\phi$ and the normalized curvature form $i\partial\overline{\partial}\phi/2\pi = dd^c\phi$ (where $d^c := i(-\partial + \overline{\partial})/4\pi$) represents the first Chern class $c_1(L)$ of L in the second real de Rham cohomology group of X. The curvature form of a smooth weight is said to be *positive* at the point x if the local Hermitian matrix $(\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j})$ is positive definite at the point x (i.e. $dd^c\phi_x > 0$). This means that the curvature is positive when $\phi(z)$ is strictly *plurisubharmonic (spsh)* i.e. strictly subharmonic along local complex lines.

More generally, a weight ψ on L is called (possibly) *singular* if $|\psi|$ is locally integrable. Then the curvature is well-defined as a $(1,1)-$current on X. The curvature current of a singular metric is called *positive* if ψ may be locally represented by a plurisubharmonic function and ψ will then simply be called a *psh weight.*

Further fixing an Hermitian metric two-form ω on X with associated volume form ω_n gives a pair (ϕ, ω_n) that will be called a weighted measure. It induces an inner product on the space $H^0(X, L)$ of holomorphic global sections of L by declaring

[1] general references for this section are the books [31, 26].

$$\|\alpha\|_\phi^2 := \int_X |\alpha|_\phi^2 \, \omega_n, \tag{1.14}$$

The corresponding Hilbert space will be denoted by $\mathcal{H}(X, L)$ and its Bergman kernel by $K(x, y)$, which is a section of the pulled back line bundle $L \boxtimes \overline{L}$ over $X \times \overline{X}$ (see section 5).

The Hermitian line bundle (L, ϕ) over X induces, in functorial way, Hermitian line bundles over all products of X (and its conjugate $\overline{X}$) and we will usually keep the notation ϕ for the corresponding weights. For example, we will write

$$|K(x,y)|_\phi^2 := |K(z,w)|^2 \, e^{-\phi(z)} e^{-\phi(w)}$$

where the right hand side is strictly speaking only defined when both x and y are contained in an open set U where L has been trivialized as above. When studying asymptotics we will replace L by its k th tensor power, written as kL in additive notation. The induced weight on kL may then be written as $k\phi$. A subindex k will indicate that the object is defined w.r.t the weight. $k\phi$ on kL for ϕ a fixed weight on L.

Regularity assumptions. A weighted measure (ϕ, μ) will be called *strongly regular* if the weight ϕ is locally $\mathcal{C}^{1,1}$-smooth (i.e. it is differentiable and all of its first partial derivatives are locally Lipschitz continuous) and $\mu = \omega_n$ is the volume form of a continuous metric ω on X. Moreover, if $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$, where is $\mathbb{P}^n$ the complex projective space, viewed as a compactification of its affine piece $\mathbb{C}^n$, then we also allow ω_n to be defined by the Lebesgue measure on $\mathbb{C}^n$ as long as the corresponding weight *function* $\phi(z)$ on $\mathbb{C}^n$ has super logarithmic growth (formula 2.4 below) with $\phi \in \mathcal{C}^{1,1}(\mathbb{C}^n)$. Occasionally, we will also consider weighted measures (ϕ, μ) which are *weakly regular* in the following sense. The weight ϕ is assumed to be continuous on a compact set E, which is locally non-pluripolar and such that the equilibrium weight of (E, ϕ) (see remark 3.1) is continuous on the complement of an analytic subvariety of X. As to the measure μ it is assumed to be supported on E and satisfying a Bernstein-Markov property with respect to the weighted set (E, ϕ) (saying that $\rho_k^{(1)}$ has subexponential growth in k; see [13]). If $\phi(z)$ has super logarithmic growth as above we only assume that E is a closed (but possibly unbounded) subset of $\mathbb{C}^n$.

Probability notation. Given a probability space (Y, γ), i.e. a measure space where $\gamma(X) = 1$, a measurable function $\mathcal{N}$ on (Y, γ) is called a *random variable.* Its integral w.r.t to Y is denoted by $\mathbb{E}(\mathcal{N})$ and called the *expectation* of $\mathcal{N}$. Recall also that if $\mathcal{N}$ takes values in a space Z then the pushforward of γ under $\mathcal{N}$ is called the *law of $\mathcal{N}$ on Z.* A subindex k will indicate that the object is defined w.r.t. the probability measure on $Y = X^{N_k}$, defined by the density 1.3 induced by a weighted measure (ϕ, μ).

Occasionally, we will also consider the probability measures defined by the Bergman kernels $K_{k\phi+\phi_F}$ associated to a sequence of Hermitian line bundles $(kL+F, k\phi+\phi_F)$ (and a fixed reference measure μ) and we will then write $\mathbb{E}=\mathbb{E}_{k\phi+\phi_F}$ etc.

2. Examples

2.1. **From projective space to orthogonal polynomials.** It is a classical fact that $\mathbb{C}^n$ is compactified by the complex projective space $X := \mathbb{P}^n$. Let L be the hyperplane line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$. Then $H^0(X, kL)$ is the space of all complex homogeneous polynomials of total degree k in $\mathbb{C}^{n+1}$, which is isomorphic to the vector space $\mathcal{H}_k(\mathbb{C}^n)$ of all polynomials in $\mathbb{C}^n$ of total degree at most k. Indeed, fix a global holomorphic section s of $\mathcal{O}(1)$, whose zero-set is $\mathbb{P}^n - \mathbb{C}^n$, the" hyper plane at infinity". Then any section s_k of $L^{\otimes k}$ over the open subset $U := \mathbb{C}^n$ may be written as

$$s_k(z) = p_k s^{\otimes k}$$

where p_k is in $\mathcal{H}_k(\mathbb{C}^n)$ (concretely, this amounts to "dehomogenizing" s_k). Moreover, the point-wise norms with respect to a metric on $k\mathcal{O}(1)$ induced by a given locally bounded metric h on $\mathcal{O}(1)$ become

$$|s_k(z)|^2_{h^{\otimes k}} = |p_k(z)|^2 e^{-k\phi(z)} \tag{2.1}$$

for some function $\phi(z)$ on $\mathbb{C}^n$, that we will call the *weight function.* As is well-known, this gives a correspondence between locally bounded metrics h on $\mathcal{O}(1)$ and weight functions $\phi(z)$ of the form

$$\phi(z) = \phi_{FS}(z) + u(z) := \ln(1+|z|^2) + u(z), \tag{2.2}$$

where u is a locally bounded function on $\mathbb{C}^n$. In particular, a subclass of weights corresponding to *smooth* metrics on $\mathcal{O}(1)$ are obtained by taking $u \in \mathcal{C}^\infty_c(\mathbb{C}^n)$. Note that the metric h_{FS} corresponding to $\phi_{FS}(z)$ is the Fubini-Study metric on $\mathcal{O}(1)$ which is characterized (up to a constant) by its $SU(n)-$invarians. Its (normalized) curvature form $\omega_{FS} := dd^c\phi_{FS}$ is the called the Fubini-Study metric on $\mathbb{P}^n$ and a simple calculation shows that the corresponding volume form is given by

$$(\omega_{FS})_n := (dd^c\phi_{FS})^n/n! = e^{-(n+1)\phi_{FS}}(\frac{i}{2})^n dz \wedge d\bar{z}$$

where $(\frac{i}{2})^n dz \wedge d\bar{z}$ denotes the Lebesgue measure on $\mathbb{C}^n$. The global norm of s_k induced by the weighted measure $(\phi, (\omega_{FS})_n)$ may hence be represented as

$$\|s_k\|^2_{(\phi,\omega_{FS})} := \int_{\mathbb{C}^n} |p_k(z)|^2 e^{-k\psi(z)} (\omega_{FS})_n. \tag{2.3}$$

Alternatively, the weight ϕ itself induces a measure $e^{-(n+1)\phi(z)}(\frac{i}{2})^n dz\wedge d\bar{z}$. The corresponding norm is hence given by

$$\|s_k\|^2_\phi := \int_{\mathbb{C}^n} |p_k(z)|^2 e^{-(k+n+1)(\phi(z))}(\frac{i}{2})^n dz\wedge d\bar{z}$$

Note that the the contribution from the factor $e^{-(n+1)\phi}$ makes sure that the integrals are finite.

2.1.1. *The setting of super logarithmic growth.* A variant of the previous setting arises if one insists on using the Lebesgue measure as the integration measure defining the norms in 2.3. Then $\phi(z)$ has to have slightly larger growth than in formula 2.2 in order to get finite norms. More precisely, we then assume that ϕ has super logarithmic growth in the sense that

$$\phi(z) \geq (1+\epsilon)\ln|z|^2, \text{ when } |z| >> 1 \tag{2.4}$$

for some positive number ϵ. It should be emphasized that such a weight ϕ does hence *not* correspond to a locally bounded metric h on $\mathcal{O}(1)$. But as shown in [8] a slight modification of the arguments apply to this super logarithmic setting, as well. The key point is that the growth condition 2.4 forces the corresponding equilibrium measure to be compactly supported in $\mathbb{C}^n$. The model case is when $\phi(z) = |z|^2$. Then the equilibrium measure is (up to a multiplicative constant) the Lebesgue measure on the unit ball.

The most general setting is the following one.

2.1.2. *The weakly regular setting and orthogonal polynomials.* The basic case of this setting appears when the integration measure in 2.3 is replaced by a measure μ which is absolutely continuous with respect to the $\mathbb{C}^n$ (or $\mathbb{R}^n$)-Lebesgue measure and supported on a domain E with smooth boundary in $\mathbb{C}^n$ (respectively in $\mathbb{R}^n$). Hence, this setting contains the classical (weighted) setting of multivariate orthogonal polynomials on domains in $\mathbb{C}^n$ and $\mathbb{R}^n$. Even more generally, $\mathbb{R}^n$ could be replaced by any maximally totally real subset of $\mathbb{C}^n$, for example the unit torus.

2.1.3. *The Vandermonde determinant and the transfinite diameter.* Fix a base (s_α) in $\mathcal{H}_k(\mathbb{C}^n)$ of multinomials (which is is hence orthonormal w.r.t. the weighted measure $(\mu_{T^n}, 0)$, where μ_{T^n} is the invariant probability measure on the unit-torus). The corresponding determinant $\det_{1\leq i,j\leq N_k}(s_i(z_i))_{i,j}$ appearing in formula 1.8 is classically called the *Vandermonde determinant* and is usually denoted by $\Delta(z_1,...,z_{N_k})$. Hence, the probability density 1.3 may, using formula 1.9, then be written as

$$|\Delta(z_1,...,z_{N_k})|^2 e^{-k(\phi(z_1)+...+\phi(z_{N_k})}/\mathcal{Z}_{k\phi} \tag{2.5}$$

w.r.t. the Lebesgue measure in $\mathbb{C}^n$. For $n > 1$ $\Delta(z_1, ..., z_{N_k})$ is a very complicated polynomial expression in $z_1, ..., z_{N_k}$. But for $n = 1$ it factorizes so that

$$|\Delta(z_1, ..., z_{N_k})|^2 = \left(\prod_{1 \leq i < j \leq k+1} |z_i - z_j| \right)^{2/k(k-1)}$$

Finally, it should be pointed out that in this setting the limit of $(\mathcal{Z}_{k\phi})^{1/k^{n+1}}$ coincides (after suitable normalization) with the multivariate weighted transfinite diameter of $(\mathbb{C}^n, \phi)$ [12, 18], whose definition goes back to the work of Leja in the fifties.

2.2. **Normal random matrices, free fermions and the Coulomb gas.** In this section we will set $n = 1$ (so that $N = k + 1$) and take the measure on $\mathbb{C}$ to be Lebesgue measure and ϕ a weight function with super logarithmic growth. References concerning this section may be found in the survey [62]. See also [28] for a similar periodic setting, where $\mathbb{C}$ is replaced by the torus $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ with a constant curvature line bundle.

2.2.1. *Random normal matrices.* Consider the set of all normal matrices $\mathcal{M}_N := \{M \in gl(N, \mathbb{C}) : [M, M^*] = 0\}$ as a Riemannian subvariety of the space $gl(N, \mathbb{C})$ of all complex matrices of rank N equipped with the Euclidean metric. A given weight function ϕ of super logarithmic growth induces the following probability measure on $\mathcal{M}_N$

$$e^{-N\mathrm{Tr}(\phi(M))} dV_{\mathcal{M}_N} / \mathcal{Z}'_{N\phi} \tag{2.6}$$

where $dV_{\mathcal{M}_N}$ is the Riemannian volume measure of $\mathcal{M}_N$ and $\mathcal{Z}'_{N\phi}$ is a normalizing constant (usually called the partition function of the corresponding matrix model). Under the map which associates the (ordered) eigenvalues $(z_1, ..., z_N)$ to a matrix M the probability measure 2.6 is pushed forward to a probability measure on $\mathbb{C}^N$ which turns out to coincide with 2.5 (with $n = 1$) times Lebesgue measure. The corresponding correlation functions $\rho_k^{(m)}$ are hence usually called *eigenvalue correlation functions* in this context. It should also be pointed out that the correlation functions corresponding to the weighted set (ϕ, μ) where μ is the invariant measure supported on $\mathbb{R}$ (or the unit-circle T) coincide with eigenvalue correlation functions for random *Hermitian* (or unitary) matrices, weighted by ϕ, which have been extensively studied (cf. [22, 39, 49] and references there in).

2.2.2. *Free fermions.* The weighted polynomials $\Psi_{+,m} := z^m e^{-k\phi(z)/2}$ where $m = 0, ..., k$ each represent the quantum state of a single spin 1/2 quantum particle (=fermion) confined to a plane subject to a magnetic field B perpendicular to the plane, where the value of B at the point z is $\frac{i}{2\pi} k \frac{\partial^2 \phi(z)}{\partial z \partial \bar{z}}$ in suitable units. Moreover, the states form a linearly independent set in the lowest possible energy level (=ground state). More precisely, this latter fact means that $\Psi_{+,m}$ is an eigenvector of finite

norm with eigenvalue 0 of the Pauli operator, which in complex notation may be written as

$$(\overline{\partial}_{k\phi}+\overline{\partial}_{k\phi}^{*})^{2}\Psi_{+,m}=0,$$

where $\overline{\partial}_{k\phi}$ intertwines the space of spin *up* and spin *down* particles

$$\overline{\partial}_{k\phi}=\overline{\partial}+\frac{k}{2}\overline{\partial}\phi\wedge:\quad \Omega^{0,0}(\mathbb{C})\rightarrow\Omega^{0,1}(\mathbb{C})$$

and $\overline{\partial}_{k\phi}^{*}$ is its formal adjoint (the corresponding real "vector potential" is given by $kA:=\frac{1}{2}k(\overline{\partial}\phi-\partial\phi)$). Hence, the particle state $\Psi_{+,m}$ is said to have spin *up*, since it has no spin down component in $\Omega^{0,1}(\mathbb{C})$ (defined is the space of element of the form $g(z)d\bar{z}$). The corresponding many particle state of N free (=non-interacting) fermions, should according to the postulates of quantum mechanics for fermions be anti-symmetric under an exchange of two single particle states Ψ_{m}. Hence, it is represented by the (Slater) determinant $\Psi(z_{1},...,x_{N}):=\det(\Psi_{+,i}(z_{j}))_{i,j}$. In particular, the corresponding probability amplitude coincides (after normalization) with the density 1.9.

2.2.3. *The coulomb gas.* Writing the probability density 1.3 in the form 1.9 identifies it with a Gibbs (Boltzmann) density at temperature $T=1$ (in suitable units)

$$\frac{e^{-\frac{1}{T}E_{k\phi}(z_{1},...,z_{N})}}{\mathcal{Z}_{k\phi}}, \tag{2.7}$$

describing an ensemble of N equal electric unit charges interacting by the Coulomb potential and subject to an exterior potential $k\phi$. Equivalently, one may rescale and write $E_{k\phi}=k^{2}E'_{k}$. Then the large k limit is the limit when the number of particles k tends to infinity in such a way that their total charge and the exterior potential are independent of k, but the temperature T is inversely proportional to k^{2}. As a consequence it is, at least heuristically, clear that the dominating contribution to 2.7 comes from configurations minimizing the total energy. Asymptotically such configurations equidistribute on the (weighted) equilibrium measure, which in this case may be defined as the measure minimizing the corresponding limiting energy functional, called the weighted logarithmic energy [52].

2.3. **Free fermions and bosonization on complex manifolds.** In the physics litterature the correspondence between free fermions in the plane and an interacting Coulomb gas is usually referred to as the *plasma analogy* (see [28] and references therein) (or rather a *one component* plasma, OCP, in the sense that all charges are equal). This analogy is also closely related to the well-known phenomena of *bosonization* in two real dimensions, where a system of *free fermions* is transformed to a system of *interacting bosons.* For a geometric treatment of bosonization on compact Riemann surfaces (which goes far beyond the scope of the present paper) see [19]. One of the aims of the present paper is to develop a similar plasma analogy (and hence bosonization) in higher dimensions

on complex manifolds and study the limit when the field strength is scaled by a large number k.

As is well-known the role of the spin 1/2 particles in the plane above is in higher dimensions played by sections of the spinor bundle $\mathcal{S}(X)$ over a spin manifold X (see [4] and [59] for the Euclidian case). More over, according to the Yang-Mills gauge principle (with structure group $U(1)$) the role of the magnetic vector potential A is played by a connection on an Hermitian line bundle L over X and the field by i times its curvature F_A. In particular, when (X, ω) is a complex Hermitian manifold and (L, ϕ) is an Hermitian holomorphic line bundle one obtains complexified spinors coupled to the corresponding canonical Chern connection (with $iF_A = dd^c\phi$) as

$$\mathcal{S}(X, L)_{\mathbb{C}} = \Omega^{0,even}(X, L) \oplus \Omega^{0,odd}(X, L)$$

where the first component (of spin up) is the space of all $(0, q)-$ forms with values in L for q even etc. The role of the Pauli operator is played by the Dolbeault Laplacian, which is the square of the Dirac operator $\overline{\partial} + \overline{\partial}^{*,\phi}$

When L is ample the Kodaira vanishing theorem gives that, for large k, the null space of the Dolbeault Laplacian on kL is precisely $H^0(X, kL)$. In particular, the ground state of fermions is spin-polarized. A many particle state of a maximal number of free fermions is hence given as an element of the top exterior power $\bigwedge^{N_k} H^0(X, kL)$. Since this space has complex dimension one any of its element may be obtained as a Slater determinant of the form 1.8. Hence, the corresponding probability amplitudes are all given by the probability density 1.9. Note also that replacing kL with $kL + F$, where F is a given Hermitian holomorphic line bundle amounts, in terms of spin geometry, to changing the spin$\mathbb{C}-$structure [4].

Finally, a remark of a rather speculative nature:

Remark 2.1. By way of comparison with [19] it seems natural to interpret an element ψ of the space of all psh weights on L as a *scalar boson field* on X and the functional $I_\phi[\psi]$ in remark 7.1 as an action on this space. It would be interesting to try to make sense of the corresponding functional integral on the space of all weights on L with minimal singularities, using for example some zeta function regularization. This would lead to a generalization of the conformal field theory setting in [19] in one complex dimension to a "holomorphic field theory" on a complex manifold of arbitrary dimension. By Legendre duality this problem should be closely related to finding a large deviation principle for the random measure 1.6 (compare remark 3.7 and remark 7.1).

3. The pluripotential equilibrium measure

Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X. Given a continuous weight ϕ on L the corresponding "equilibrium weight" ϕ_e is defined as the envelope

$$\phi_e(x) := (P\phi)(x) := \sup\left\{\widetilde{\phi}(x) : \widetilde{\phi} \in \mathcal{L}_{(X,L)}, \, \widetilde{\phi} \leq \phi \text{ on } X\right\}. \tag{3.1}$$

where $\mathcal{L}_{(X,L)}$ is the class consisting of all (possibly singular) psh weights on L. Then ϕ_e is also in the class $\mathcal{L}_{(X,L)}$ [33] The notation $\phi_e = P\phi$ will occasionally be used to indicate that ϕ_e is the image of ϕ under the *psh projection operator* P. The Monge-Ampere measure $(dd^c\phi_e)^n/n!$ is well-defined on the open set

$$U(L) := \{x \in X : \phi_e \text{ is bounded on } U(x)\},$$

where $U(x)$ is some neighborhood of x (see [3, 42, 33] for the definition of the Monge-Ampere measure of a locally bounded metric or plurisubharmonic function). *The equilibrium measure* (associated to the weight ϕ) is now defined as

$$\mu_{\phi_e} := 1_{U(L)}(dd^c\phi_e)^n/n! \tag{3.2}$$

and is hence a positive measure on X. Next, consider the following "incidence set":

$$D := \{\phi_e = \phi\} \subset X, \tag{3.3}$$

As is well-known the equilibrium measure μ_ϕ is supported on D.

Remark 3.1. If E is a closed locally non-pluripolar set of X the equilibrium weight of the weighted set (E, ϕ) is defined be replacing X with E in the definition 3.1. It will be denoted by $P_E\phi$ and assumed to be upper semi-continuous and hence psh. The equilibrium measure of (E, ϕ) is then defined as the Monge-Ampere measure of $P_E\phi$ and is supported on E. See [12] for the general setup.

The following regularity theorem obtained in [9] (see Theorem 3.4 and Remark 3.6 there) shows that if ϕ is class $\mathcal{C}^{1,1}$ on X, than ϕ_e is in the class $\mathcal{C}^{1,1}$ on the complement of an analytic subvariety $\mathbb{B}_+(L)$ (the augmanted base locus of L). See [9, 43] for the precise definition of $\mathbb{B}_+(L)$. Here we just point out that $\mathbb{B}_+(L)$ is a proper subvariety precisely when L is big and is empty precisely when L is ample.

Theorem 3.2. *Suppose that L is a big line bundle and that the given metric ϕ on L is in the class $\mathcal{C}^{1,1}$). Then*

(a) ϕ_e is in the class $\mathcal{C}^{1,1}$ on $X - \mathbb{B}_+(L)$.

(b) The Monge-Ampere measure of ϕ_e on $X - \mathbb{B}_+(L)$ is absolutely continuous with respect to any given volume form and coincides with the corresponding L^∞_{loc} $(n,n)-$form obtained by a point-wise calculation:

$$(dd^c\phi_e)^n = \det(dd^c\phi_e)\omega_n \tag{3.4}$$

(c) the following identity holds almost everywhere on the set $D - \mathbb{B}_+(L)$, where $D = \{\phi_e = \phi\}$:

$$\det(dd^c\phi_e) = \det(dd^c\phi) \tag{3.5}$$

More precisely, it holds for all points in the complement of the augmented base locus $\mathbb{B}_+(L)$ where the second order jet $(\phi_e - \phi)^{(2)}$ exists and vanishes, i.e. pointwise on

$$(X - \mathbb{B}_+(L)) \cap \{(\phi_e - \phi)^{(2)} = 0\} \tag{3.6}$$

(d) Hence, the following identity between measures on X holds:

$$n!\mu_{\phi_e} = 1_{X-\mathbb{B}_+(L)}(dd^c\phi_e)^n = 1_D(dd^c\phi)^n = 1_{D\cap X(0)}(dd^c\phi)^n \tag{3.7}$$

Definition 3.3. The set in formula 3.6 above is called the *bulk (of (X, ϕ))* and its *pseudo-interior* is the set of all points x such that x has a neighborhood whose intersection with the complement of the bulk has measure zero (w.r.t ω_n).

The definition is made so that, in the bulk, the density of the equilibrium measure (w.r.t. ω_n) exists and is equal to $\det(dd^c\phi)$ and vanishes a.e. on the complement of the bulk. The next proposition gives a simple sufficient criterion for a point to be in the pseudo-interior of the bulk. It applies, for example, to points where μ_{ϕ_e} has a continuous positive density.

Proposition 3.4. *Let x be a point such that x has a neighborhood U where the equilibrium measure $\mu_{\phi_e}$is represented by a density which is bounded from below by a positive constant and where ϕ is in the class $\mathcal{C}^2$. Then x is in the interior of $D \cap X(0)$ and in particular in the pseudo-interior of the bulk.*

Proof. Take x in U as above. Assume for a contradiction that x is in the open set $X - D$. Then there is a neighborhood of x, contained in $X - D$, where $\mu_{\phi_e} > \delta\omega_n$ as measures. But this contradicts the fact that $\mu_{\phi_e} = 0$ on $X - D$ [33]. Since, x was an arbitrary point in U this means that U is contained in the interior of D, i.e. $\phi_e = \phi$ on U. By the regularity assumption on ϕ it follows that $(dd^c\phi)^n > 0$ point-wise on U. But since ϕ is psh on D [9] this forces $dd^c\phi > 0$ on U. All in all this means that U is contained in the interior of $D \cap X(0)$, which by definition is contained in the pseudo-interior of the bulk. □

Remark 3.5. Even in the classical one-dimensional case where $(X, L) = (\mathbb{P}^1, \mathcal{O}(1))$ and ϕ is smooth, the equilibrium weight may not have second derivatives at some points. In fact, when ϕ is radial this happens "generically" [9]. Moreover, there seem to be essentially no general results concerning the regularity of the incidence set D or the bulk. It would be interesting to find conditions under which the bulk is a domain with suitable regularity properties. For example, if X is one-dimensional and ϕ real-analytic it seems plausible that the bulk is a domain whose topological boundary consists of piece-wise real-analytic curves.

3.1. **The free energy functional.** Define the following bifunctional on the space of weights:

$$\mathcal{E}[\psi,\psi'] := \sum_{j=0}^{n} \int_X (\psi-\psi')(dd^c\psi)^j \wedge (dd^c\psi')^{n-j}/(n+1)!, \tag{3.8}$$

Then it is known since the work of Aubin, Mabuchi and others, at least in the case when ψ is smooth, that the differential of $\mathcal{E}$ at ψ is represented by the Monge-Ampere measure of ψ, i.e.

$$\frac{d\mathcal{E}[\psi+tu,\psi']}{dt}_{t=0} = \int_X u(dd^c\psi)^n/n!$$

Following the definition of the energy functional in [12] we define the *free energy (relative a reference weight ϕ_0)* by

$$\mathcal{F}_\infty[\phi] := \mathcal{E}[P\phi, P\phi_0],$$

where we recall that P is the psh projection operator that maps a weight to its equilibrium weight defined above. One of the main results of [12] then says that $\mathcal{F}_\infty[\phi]$ is a primitive of the equilibrium measure, i.e. if u is continuous then

$$\frac{d\mathcal{F}_\infty[\phi+tu]}{dt}_{t=0} = \int_X u\mu_{\phi_e} \tag{3.9}$$

Next, we will consider second derivatives, as well. Fix a family ϕ_t of weights on L where $t \in \Delta$ (the unit-disc in $\mathbb{C}$) which will be assumed to be locally $\mathcal{C}^{1,1}-$smooth in the $n+1$ complex variables (z,t). Such a family ϕ_t will simply be a $\mathcal{C}^{1,1}-$ *family.* Equivalently, the family ϕ_t of weights on $L \to X$ may be identified with a $\mathcal{C}^{1,1}-$smooth weight Φ on the pulled-back line bundle over $\Delta \times X$ under the projection onto X. We will identify t with the projection onto Δ.

Proposition 3.6. *Let ϕ_t be a curve of weights as above with $\phi_0 = \phi$ such that $\frac{\partial}{\partial t}\phi_t$ is, for each fixed t, supported in the union of the pseudo-interior of the bulk and $X - D$. Then the following holds for $|t|$ small: $\frac{\partial}{\partial t}P(\phi_t) = \frac{\partial}{\partial t}\phi_t$ in the bulk and $\frac{\partial}{\partial t}P(\phi_t) = 0$ in $X-D$.* Moreover,

$$d_t d_t^c \mathcal{F}_\infty[\phi_t] = t_*(dd^c\Phi)^{n+1}/(n+1)! \tag{3.10}$$

if $\frac{\partial}{\partial t}\phi_t$ is supported in the pseudo-interior of the bulk, for each fixed t.

Proof. We will denote by B the pseudo-interior of the bulk. Denote by E a ("small") compact set containing the support of $\frac{\partial}{\partial t}\phi_t$ in X for small t. Let us first show that $P\phi_t = \phi_t$ in the bulk, i.e. B. To this end let ψ_t be the weight on L defined as ϕ_t on $E \cap B$ and as $P\phi$ on $X - E \cap B$. Then ψ_t is continuous since it coincides with ϕ_t on some neighborhood U of E. In particular, ψ_t is psh on U for t small (since $dd^c\phi_0$ has a positive lower bound in the sense of currents there). But then ψ_t is psh on all of X (since $P\phi$ is). Moreover, by construction $\psi_t \leq \phi_t$ on X. Hence, the extremal definition of $P\phi_t$ gives $P\phi_t \geq \psi_t$, forcing $P\phi_t = \phi_t$ in B as was to be shown.

Next, we will show that $P\phi_t = P\phi_0$ in $X - D$. It is enough to prove this on any fixed compact subset K in $X - D$. By assumption $P\phi_0 < \phi_0$ on D, forcing $P\phi_0 < \phi_t$ on K for t sufficiently small. Hence, $P\phi_0 \leq P\phi_t$ by the extremal definition of $P\phi_t$. Finally, by the continuity of the operator P [12] we also have $P\phi_t < \phi_0$ on K so that $P\phi_t \leq P\phi_0$. Hence, $P\phi_0 = P\phi_t$ on K as was to be shown.

Finally, since, as shown above $P\phi_t = \phi_t$ on D and $P\phi_t < \phi_t$ (by continuity) on $X - D$ it follows that $D_t = D$. Formula **??** then follows by differentiating the right hand side in formula 3.9:

$$\frac{\partial^2 \mathcal{F}_\infty[\phi_t]}{\partial t \partial \bar{t}} = \frac{\partial}{\partial t} \int \frac{\partial \phi_t}{\partial \bar{t}} (dd^c P(\phi_t)^n / n! =$$

$$= \int \frac{\partial^2 \phi_t}{\partial t \partial \bar{t}} (dd^c P(\phi_t)^n / n! - \frac{\partial \phi_t}{\partial \bar{t}} dd^c \frac{\partial \phi_t}{\partial t} \wedge (dd^c P(\phi_t))^{n-1} / (n-1)!$$

Since by assumption $P\phi_t := \phi_t$ an the support of $\frac{\partial \phi_t}{\partial t}$ a partial integration in the second term above finally proves 3.11. $\square$

In particular, if $\phi_t = \phi + (\mathrm{Re} t) u$, and where u is supported in the pseudo-interior of the bulk, then we get

$$\frac{d^2 \mathcal{F}_\infty[\phi + tu]}{d^2 t}_{t=0} = - \|du\|^2_{dd^c \phi} \tag{3.11}$$

On the other hand, as shown in [12] there are examples where the second derivatives above do not exist, due to the fact that u is not supported in the interior of the bulk (or the complement of D).

Remark 3.7. As shown in [12] the equilibrium weight ϕ_e can also, up to an additive constant, be charactarized as the weight which realizes the infimum of the following funtional on the space of weights on L with minimal singularities:

$$I_\phi[\psi] := \mathcal{E}[\psi, \phi] + \int MA(\psi)(\phi - \psi) \tag{3.12}$$

where the infinum is hence equal to $C_\phi := \mathcal{E}[\phi_e, \phi]$. More generally, fixing a reference weight ϕ_0 the the equilibrium weight ϕ_e realizes the infimum of

$$I_{\phi,\phi_0}[\psi] := \mathcal{E}[\psi, \phi_0] + \int MA(\psi)(\phi - \psi)$$

since the functionals only differ by the constant $\mathcal{E}[\phi_0, \phi]$. The equilibrium measure is hence the Monge-Ampere measure of any minimizer of any of the previous functionals. For possible future applications to large deviations it is convenient to reformulate the previous extremal problem in terms of measures μ satifying $\int_X \mu = \mathrm{Vol}(L)$, which we assume equals one (otherwise we just rescale the functionals). To this end assume for simplicity that L is ample. Then, by the work of Guedj-Zeriahi [34], there is a certain class $\mathcal{E}_1(X)$ of measures μ such that the Monge-Ampere equation

$$MA(\psi) = \mu$$

can be solved uniquely modulo constants and such that the solution is in the dual class $\mathcal{E}^1(X)$. .[2]Moreover, the functional I_ϕ is well-defined (and finite) on $\mathcal{E}^1(X)$. Denote any such solution by ψ_μ. Since, the functional $I_\phi[\psi]$ is invariant under $\psi \mapsto \psi + c$ this means that, using the work of Guedj-Zeriahi

$$I'_\phi[\mu] := I'_\phi[\psi_\mu] \tag{3.13}$$

is a well-defined functional on $\mathcal{E}_1(X)$. Now, if ϕ is, say smooth, then $MA(\phi_e)(= \mu_{\phi_e})$ is in $\mathcal{E}_1(X)$ and hence the infimum of 3.13 is attained at the corresponding equilibrium measure μ_{ϕ_e} and only there. In the light of the usual thermodynimical formalism it seems natural to also define the *entropy functional* on $\mathcal{E}_1(X)$:

$$S[\mu] := \mathcal{E}(\psi_\mu, \phi_0) - \int \mu(\psi_\mu - \phi_0)$$

w.r.t. a reference psh weight ϕ_0. The point is that $S[\mu]$ is then the Legendre transform of the free energy functional $I_{\phi,\phi_0}[\psi]$.

4. Weighted L^2−estimates for $\overline{\partial}$

In this section we will generalize, by refining the results in [9], some well-known estimates for the $\overline{\partial}-$operator concerning psh weights to more general weights. More precisely, we will assume that ϕ is a locally $\mathcal{C}^{1,1}-$smooth weight on the line bundle L over X. When $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$ we also allow weights corresponding to a weight function $\phi(z)$ in $\mathbb{C}^n$ with super logarithm growth (see section 2). But for simplicity we do not consider the latter situation in the proofs. The simple modifications needed follow precisely as in the appendix in [8].

We will denote by K_X the canonical line bundle of L, whose smooth sections are $(0, n)-$forms on X. As a consequence a weight ϕ on L induces, even without choosing a volume form ω_n on X, an L^2-norm on sections u of $L + K_X$ that we will write as

$$\|u\|^2_\phi := \int_X |u|^2 \, e^{-\phi}$$

In the statement of the following theorem, we will use the fact that $dd^c\phi$ defines a positive form with locally bounded coefficients in the bulk (by the very definition of the bulk).

Theorem 4.1. *Let L be a big line bundle and ϕ a $\mathcal{C}^{1,1}-$smooth weight. Then for any $\overline{\partial}-$closed $(0,1)-$form g with values in $L+K_X$ and supported in the pseudo-interior of the bulk, there is a smooth section u with values in $L + K_X$ such that*

$$\overline{\partial}u = g$$

[2]The class $\mathcal{E}^1(X)$ consists of psh weights on L with gradient in $L^2_{loc}(X)$ and a measure μ is in $\mathcal{E}_1(X)$ iff all elements in $\mathcal{E}^1(X)$ are locally integrable w.r.t μ.

and

$$\int_X |u|^2 e^{-\phi} \leq \int_X |g|^2_{dd^c\phi} e^{-\phi}. \tag{4.1}$$

In particular, the previous estimate holds for any u such that u is orthogonal to $H^0(X, L+K_X)$ (w.r.t the weight ϕ).

Proof. Let ψ denote a general psh weight on L. By theorem 5.1 in [25] the theorem holds with ϕ replaced by a (possibly singular) psh weight ψ if $dd^c\phi$ is replaced with the absolutely continuous part $(dd^c\psi)_c$ of the Lebesgue decomposition of the positive form $dd^c\psi$. More precisely,

$$\int_X |u|^2 e^{-\psi} \leq \int_X |g|^2_{(dd^c\psi)_c} e^{-\psi} \tag{4.2}$$

as long as the r.h.s is finite. Now set $\psi = \phi_e$, the equilibrium weight corresponding to ϕ. Since g is supposed to be supported in the bulk, the regularity Theorem 3.2, gives

$$\int_X |g|^2_{(dd^c\phi_e)_c} e^{-\phi_e} = \int_X |g|^2_{(dd^c\phi)} e^{-\phi}$$

and since g is, in fact, supposed to be supported in the *pseudo-interior* of the bulk the latter integral is finite. Finally, using that $\phi_e \leq \phi$ on *all* of X finishes the proof of the estimate 4.1. The last statement of the theorem now follows since the estimate 4.1 in particular holds for the solution which minimizes the corresponding L^2-norm. □

The previous theorem is a generalization to non-psh weights ϕ of the fundamental result of Hörmander-Kodaira. In turn, the next theorem is a generalization to non-psh weights of a refinement of the Hörmander-Kodaira estimate which goes back to a twisting trick in the work of Donelly-Fefferman. See [24, 44] for an analogous result concerning psh weights in $\mathbb{C}^n$.

Theorem 4.2. *Let L be a big line bundle, ϕ a $\mathcal{C}^{1,1}-$smooth weight on L and v a smooth function on E such that dv is supported in the interior of the bulk of (X, ϕ) and*

$$(i)\ \left|\overline{\partial}v\right|^2_{dd^c\phi} \leq 1/8 \quad (ii)\ dd^c v \geq -dd^c\phi/2$$

there. Then

$$\int_X |u|^2 e^{-\phi_e+v} \leq 2\int_X \left|\overline{\partial}u\right|^2_{dd^c\phi} e^{-\phi_e+v}$$

for any smooth section u of $L+K_X$ orthogonal to the space $H^0(L+K_X)$, w.r.t the weight ϕ, and such that $\overline{\partial}u$ is supported in the pseudo-interior of the bulk of (X, ϕ)

Proof. By assumption

$$\langle u, h\rangle_\phi = 0, \ \ \forall h \in H^0(X, L+K_X).$$

Equivalently, writing $u_v := ue^v$,

$$\langle u_v, h\rangle_{\phi+v} = 0, \ \ \forall h \in H^0(X, L+K_X). \tag{4.3}$$

By Leibniz rule

$$\overline{\partial}u_v = (\overline{\partial}u + \overline{\partial}vu)e^v, \tag{4.4}$$

which by assumption is supported in the bulk of (X,ϕ). Hence, applying the estimate 4.2 in the proof of the previous theorem to $\psi = \phi_e + v$ gives, since by assumption *ii* $(\phi_e + v)$ is a psh weight

$$\int_X |u_v|^2 e^{-(\phi_e+v)} \leq \int_X \left|\overline{\partial}u_v\right|^2_{dd^c(\phi_e+v)} e^{-(\phi_e+v)} \leq \int_X \left|\overline{\partial}u_v\right|^2_{\frac{1}{2}dd^c\phi} e^{-(\phi+v)}$$

for *some* solution u_v of the corresponding $\overline{\partial}-$equation and hence for u_v as in formula 4.3 (we are also using that $\overline{\partial}u$ and $\overline{\partial}v$ are supported in the bulk of (X,ϕ) to replace ϕ_e with ϕ in the r.h.s). Using $\phi_e \leq \phi$, 4.4 and the "parallelogram law" then gives

$$\int_X |u|^2 e^{-\phi}e^v \leq 4\int_X (\left|\overline{\partial}u\right|^2_{dd^c\phi} + \left|\overline{\partial}vu\right|^2_{dd^c\phi})e^{-\phi_e}e^v$$

Finally, by assumption (i) in the theorem the term in the r.h.s involving $\overline{\partial}vu$ may be absorbed in the l.h.s. □

Corollary 4.3. *Let L be a big line bundle and let $k\phi+\phi_F$ be a $\mathcal{C}^{1,1}-$smooth weight on and ω_n a fixed volume form on X. Let E be a given compact subset of the interior of the bulk. Then there is a constant C (depending on E and F) such that the following holds. If ψ_k is a sequence of functions such that $d\psi_k$ is supported in the interior of the bulk of (X,ϕ) and*

$$(i)\ \left|\overline{\partial}\psi_k\right|^2_{dd^c\phi} \leq 1/C \quad (ii)\, dd^c\psi_k \leq \sqrt{k}dd^c\phi/C$$

Then

$$\left\|\Pi_k(f_k) - f_k\right\|^2_{k\phi+\phi_F+\sqrt{k}\psi_k} \leq C\frac{1}{k}\left\|\overline{\partial}f_k\right\|^2_{k\phi+\phi_F+\sqrt{k}\psi_k},$$

for any sequence f_k of smooth sections of kL, where Π_k is the Bergman projection with respect to $k\phi$ (formula 5.1 and below). Moreover, given a fixed smooth weight ϕ_{F_0} on F and a constant C' the constant C may be taken to be independent on the weight ϕ_F on F as long as $\left\|\partial^\alpha(\phi_F - \phi_{F_0})\right\|_\infty \leq C'$ for all multiindices α of total degree at most two.

Proof. Replacing L with $kL+F-K_X$, ϕ with $k\phi+\phi_F$ and v with $\sqrt{k}\psi_k$ the corollary follows from the previous theorem using standard properties of orthogonal projections. □

Proposition 4.4. *The following local estimate holds for all u which are $\mathcal{C}^1-$smooth:*

$$\sup_{|z|\leq Rk^{-1/2}} |u(z)|^2 e^{-k\phi(z)} \leq C_R k^n (\int_{|z|\leq 2Rk^{-1/2}} (|u|^2 + \frac{1}{k}\left|\overline{\partial}u\right|^2)e^{-k\phi}\omega_n) \tag{4.5}$$

Proof. This is a generalization of the uniformity statement in lemma 5.3. It is proved in essentially the same way, by replacing the mean value property of holomorphic functions used to prove lemma 5.3 by the general Cauchy formula for a smooth function u. It is also a consequence

of Gårding's inequality - see (the proof of) lemma 3.1 in [5] for a more general inequality. □

5. Asymptotics for Bergman kernels and correlations

5.1. **Bergman kernels.** Recall that $\mathcal{H}(X,L)$ denotes the Hilbert space obtained by equipping the vector space $H^0(X,L)$ with the inner product corresponding to the norm () induced by the weighted measure (ϕ,ω_n). Let (s_i) be an orthonormal base for $\mathcal{H}(X,L)$. The *Bergman kernel* of the Hilbert space $\mathcal{H}(X,L)$ may be defined as the holomorphic section

$$K_k(x,y)=\sum_i s_i(x)\otimes\overline{s_i(y)}. \tag{5.1}$$

of the pulled back line bundle $L\boxtimes\overline{L}$ over $X\times\overline{X}$. To see that is independent of the choice of base (s_I) one notes that K_k represents the integral kernel of the orthogonal projection Π_k from the space of all smooth sections with values in L onto $\mathcal{H}(X,L)$.

The restriction of K_k to the diagonal is a section of $L\otimes\overline{L}$. Hence, its point wise norm $\left|K_k(x,x)\right|_\phi (=\left|K_k(x,x)\right|e^{-k\phi(x)})$ defines a well-defined function on X that will be denoted by $\rho^{(1)}$ (and later identified with the *one point correlation function*):

$$\rho^{(1)}(x):=\sum_i \left|s_i(x)\right|^2_{k\phi}. \tag{5.2}$$

It has the following well-known extremal property:

$$\rho^{(1)}(x):=\sup\left\{\left|s(x)\right|^2_\phi:\ s\in\mathcal{H}(X,L),\ \left\|s\right\|^2_\phi\leq 1\right\} \tag{5.3}$$

Moreover, integrating 5.2 shows that $\left|K_k(x,x)\right|_\phi$ is a "dimensional density" of the space $\mathcal{H}(X,L)$:

$$\int_X \rho^{(1)}(x)\omega_n=\dim\mathcal{H}(X,L):=N \tag{5.4}$$

In section 6.1 we will consider a function on the $N-$fold product X^N that may, abusing notation slightly, be written as

$$\rho^{(N)}(x_1,...,x_N)=\det_{1\leq i,j\leq N}(K(x_i,x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))}). \tag{5.5}$$

To clarify the notation denote by $L^{\boxtimes N}$ the pulled-back line bundle on X^N with the weight induced by the weight ϕ on L. Then the base $S=(s_i)$ in $H^0(X,L)$ induces an element $\det(S)$ in $H^0(X^N,L^{\boxtimes N})$ whose value at $(x_1,...,x_N)$ is defined as the determinant

$$\det(S)(x_1,.x_N):=\det_{1\leq i,j\leq N}(s_i(x_i))_{i,j}\in L_{x_1}\otimes\cdots\otimes L_{x_N}.$$

In particular, its point-wise norm is a *function* on X^N which according to the following lemma may be locally written in the form 5.5. The lemma also shows that after division by $N!$ this function defines the density of a probability measure on X^N. Its proof is based on the following

"integrating out" property of the Bergman kernel K, which is a direct consequence of the fact that K is a projection kernel:

$$|K(x,x)|_\phi = \int_X |K(x,y)|_\phi^2 \,\omega_n(y) \tag{5.6}$$

Lemma 5.1. *The following identities hold point-wise:*

$$\det_{1\leq i,j\leq N}(K(x_i,x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))}) = |\det(S)(x_1,.x_N)|_\phi^2 .$$

Integrating gives

$$\int_{X^N} |\det(S)(x_1,.x_N)|_\phi^2 \,\omega_n^{\otimes N} = N!.$$

Proof. The identities are formal consequences of the identity 5.6, as is well-known in the random matrix literature. See for example [22]. $\square$

5.2. **Scaling asymptotics of $K_k(x,y)$ in the bulk.** In this section we fix a continuous metric ω on X. Given a point x in X we can take "normal" local coordinates z centered at x and a "normal" trivialization of L, i.e such that

$$\omega_x = \frac{i}{2}\sum_{i=1}^n dz_i \wedge \overline{dz_i} + o(1) \;\; \phi(0) = d\phi(0) = 0 \tag{5.7}$$

Moreover, if the second partial derivatives of ϕ exist at x then we may assume

$$(dd^c\phi)_x = \frac{i}{2\pi}\sum_{i=1}^n \lambda_i dz_i \wedge \overline{dz_i}$$

Hence, the λ_i are the eigenvalues of the curvature form $dd^c\phi$ at x w.r.t the metric ω and we denote the corresponding diagonal matrix by λ.

For proofs of the following elementary local consequences of the regularity properties of ϕ and ϕ_e see [8].

Lemma 5.2. *Given a point x in X and "normal" local coordinates z centered at x and a "normal" trivialization of L the following holds:*

$$|\phi(z)| \leq C\,|z|^2 , \tag{5.8}$$

C can be taken to be independent of the center x on any given compact subset of X. Moreover, if the second partial derivatives of ϕ exist at $z=0$ (the exceptional set has measure zero) then for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$(|z| \leq \delta \Rightarrow \left|\phi(z) - \sum_{i=1}^n \lambda_i\,|z_i|^2\right| \leq \epsilon\,|z|^2 \tag{5.9}$$

and for any fixed positive number R the following uniform convergence holds when k tends to infinity

$$\sup_{|z|\leq R}\left|k\phi(\frac{z}{\sqrt{k}})-\sum_{i=1}^{n}\lambda_i\left|z_i\right|^2\right|\to 0, \tag{5.10}$$

Finally, if the center x is in the bulk, then for any $\epsilon>0$, there is a $\delta>0$ such that

$$(iii)\ \ |z|\leq\delta\Rightarrow|\phi_e(z)-\phi(z)|\leq\epsilon\left|z\right|^2 \tag{5.11}$$

The next lemma only uses *local* properties of holomorphic functions and was called *local holomorphic Morse inequalities* in [5]. See [8] for the proof when the weight ϕ is merely $C^{1,1}-$smooth.

Lemma 5.3. *Fix a center x in X where the second derivatives of the weight ϕ exist and normal coordinates z centered at x. Then*

$$\limsup_k k^{-n}\rho_k^{(1)}(z/k^{1/2})\leq\det_\omega(dd^c\phi)(x).$$

Moreover, if $|z|\ \leq R$ then the l.h.s. above is uniformly bounded by a constant C_R which is independent of the center x.

Now we can prove the following *lower* bound on the 1-point correlation function in the bulk, which is a refinement of Lemma 4.4 in [9]:

Lemma 5.4. *Fix a center x in the bulk and normal coordinates z centered at x. Then*

$$\liminf_k k^{-n}\rho_k^{(1)}(z/k^{1/2})\geq\det_\omega(dd^c\phi)(x)$$

Proof. Step1: construction of a smooth extremal σ_k. Fix a center x_0 in the bulk. First note that there is a *smooth* section σ_k with values in $kL+F$ such that

(5.12)

$$(i)\lim_{k\to\infty}\frac{\left|\sigma_k\right|^2_{k\phi}(z_0/\sqrt{k})}{k^n\left\|\sigma_k\right\|^2_{k\phi+\phi_F}}=(\frac{1}{2\pi})^n\det\lambda,\ \ (ii)\left\|\overline{\partial}\sigma_k\right\|^2_{k\phi_e+\phi_F}\leq Ce^{-k/C}$$

To see this first take normal trivializations of L and F and normal coordinates z centered at x_0. Next, by scaling the coordinates z we can assume that

$$\omega_{x_0}=\frac{i}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}dz_i\wedge\overline{dz_i},\ \ (dd^c\phi)_{x_0}=\frac{i}{2\pi}\sum_{i=1}^{n}dz_i\wedge\overline{dz_i}$$

Fix a smooth function χ which is equal to one when $|z|\leq\delta/2$ and supported where $|z|\leq\delta$; the number δ will be assumed to be sufficiently small later on. Now σ_k is simply obtained as the local section with values in L^k represented by the function

$$\chi(z)e^{k(\bar{z_0}z-\frac{1}{2}\bar{z_0}z_0)}$$

close to x_0 and extended by zero to all of X. To see that (i) holds note first consider the nominator:

$$\left|\sigma_k\right|^2_{k\phi}(z_0/\sqrt{k})=1^2e^{(\bar{z_0}z_0)}e^{-k\phi(z_0/\sqrt{k})}\to 1,$$

when k tends to infinity, using 5.10. Next, write the the integrand in $k^n \|\sigma_k\|^2_{k\phi+\phi_F,}$ in the form

$$\chi(z)^2 k^n e^{-k(\left|z-z_0/\sqrt{k}\right|^2+(\phi(z)-|z|^2))}((\det\lambda)^{-1}+o(1))$$

and decompose the region of integration according to the following decomposition of the radial values:

$$[0,\delta]=[0,R/\sqrt{k}]\bigsqcup[R/\sqrt{k},\delta], \tag{5.13}$$

where R is a fixed large number. In the first region, we have by 5.10,

$$\sup_{|z|\leq R/\sqrt{k}}\left|k(\phi(z)-|z|^2)\right|\to 0$$

Hence, performing the change of variables $z=z'/\sqrt{k}$ gives

$$\lim_{k\to\infty} k^n \|\sigma_k\|^2_{k\phi+\phi_F,[0,R/\sqrt{k}]}=(\det\lambda)^{-1}\int_{[0,R]} e^{-|z'-z_0|^2}(\frac{i}{2}\sum_{i=1}^n dz'_i\wedge\overline{dz'_i})^n/n!$$

As fort the second region in 5.13 we have

$$\left|z-z_0/\sqrt{k}\right|^2+(\phi(z)-|z|^2)\geq\frac{1}{2}|z|^2 \tag{5.14}$$

for R sufficiently large. Indeed, by 5.9

$$|z|\leq\delta\Rightarrow\left|(\phi(z)-|z|^2)\right|\leq\frac{1}{4}|z|^2.$$

Moreover,

$$\left|z-z_0/\sqrt{k}\right|^2\geq\frac{1}{4}|z|^2,$$

for all k, if R is sufficiently large. Hence,

$$k^n\|\sigma_k\|^2_{k\phi+\phi_F,[R/\sqrt{k},\delta]}\leq\int_{[R/\sqrt{k},\delta]}k^n e^{-k\frac{1}{2}|z|^2}\to 0,$$

since it is the "tail" of a convergent (Gaussian) integral (using the change of variables $z=z'/\sqrt{k}$ again). Finally, letting first k and then R tend to infinity finishes the proof of (i) in 5.12.

Next, to prove (ii) in 5.12, first note that

$$\left\|\overline{\partial}\sigma_k\right\|^2_{k\phi_e+\phi_F}\leq C'\int_{\delta/2\leq|z|\leq\delta} e^{-k(\left|z-z_0/\sqrt{k}\right|^2+(\phi(z)-|z|^2)+\phi_e(z)-\phi(z)))}\omega_n(0) \tag{5.15}$$

as follows from the definition of χ. Now take δ so that, using 5.9 and 5.11 ,

$$|z|\leq\delta\Rightarrow\phi(z)+(\phi_e(z)-\phi(z))\geq|z|^2/4 \tag{5.16}$$

for δ sufficiently small. Combining 5.14 and 5.16 shows that the exponent in 5.15 is at most$-\frac{1}{4}k|z|^2$ which proves (ii) in 5.12.

Step2: perturbation of σ_k to a holomorphic extremal α_k.

This step is just a repetition (word for word) of the corresponding step in the proof of lemma 4.4 in [9]. For completeness we recall it briefly

here. Equip $kL+F$ with a "strictly positively curved modification" ψ_k of the metric $k\phi_e+\phi_F$ furnished by lemma **??** in [9]. Let $g_k=\overline{\partial}\sigma_k$ and let α_k be the following holomorphic section

$$\alpha_k := \sigma_k - u_k,$$

where u_k is the solution of the $\overline{\partial}$-equation in the Hörmander-Kodaira theorem 4.1 with $g_k=\overline{\partial}\sigma_k$. Using properties of ϕ_e on then obtains the estimate

$$\|u_k\|_{k\phi+\phi_F} \leq C\,\|g_k\|_{k\phi_e+\phi_F} \tag{5.17}$$

and then (ii) in 5.12 in the right hand side gives

$$(a)\ \|u_k\|_{k\phi+\phi_F} \leq Ce^{-k/C},\ \ (b)\ |u_k|^2_{k\phi+\phi_F}(x) \leq C'k^n e^{-k/C'},$$

where (b) is a consequence of (a) and the local holomorphic Morse inequalities **??** applied to u_k at $z=0$. Combining (a) and (b) with (i) in 5.12 then proves that (i) in 5.12 holds with σ_k replaced by the holomorphic section α_k. By the definition of $\rho_k^{(1)}$ this finishes the proof of the lemma. □

Before turning to the proof of Theorem 1.1 we also recall the following uniform estimate (which follows from lemma 5.3 precisely as in lemma 5.2 (i) in [7]):

Lemma 5.5. *Fix a center x in X and normal coordinates z and w centered at x with z,w contained in a fixed compact set. Then*

$$k^{-2n}\left|K_k(z/k^{1/2},w/k^{1/2}))\right|^2_{k\phi+\phi_F} \leq C$$

for some constant independent of the center x in X.

5.2.1. *Proof of Theorem 1.1.* Fix a point x_0 in X and take coordinates z and w centered at x and normal trivializations of L and F as in the proof of the previous lemma, inducing corresponding trivializations around (x,x) in $X\times X$. Consider the holomorphic functions $f_k(z,w)=k^{-n}K_k(k^{-1/2}z,k^{-1/2}\bar{w})$ and $f(z,w)=\det_\omega(dd^c\phi)(x_0)e^{zw}$ on the polydisc on Δ_R of radius R centered at the origin in $\mathbb{C}^{2n}$. By lemma 5.5:

$$\sup_{\Delta_R}|f_k| \leq C_R, \tag{5.18}$$

Moreover, combining the upper and lower bounds in lemma 5.3 and lemma 5.4, respectively, shows that f_k tends to f on $M:=\{(z,\bar{z})\in\Delta_R\}$. Now, by the bound 5.18 f_k has a convergent subsequence converging uniformly on Δ_R to a holomorphic function f_∞ where necessarily $f_\infty=f$ on M. But since M is a maximally totally real submanifold it follows that $f_\infty=f$ everywhere on Δ_R. Since, the argument can be repeated for any subsequence of f_k this proves the uniform convergence in the theorem. Finally, the convergence of higher derivatives is a standard consequence of Cauchy estimates.

Remark 5.6. In fact, Theorem 1.1 also follows in a more or less formal way (using the method in ref) from combining Lemma 5.3 with the the special case of Lemma 5.4 obtained by setting $z = 0$ (which was obtained in [9]). But the present method is more explicit and hence gives a better control on the convergence, which might be useful in other contexts.

5.3. **Off-diagonal decay of** $K_k(x, y)$**.**

Theorem 5.7. *Let L be a big line bundle and K_k the Bergman kernel of the Hilbert space $\mathcal{H}(kL + F)$. Let E be a compact subset of the interior of the bulk. Then there is a constant C (depending on E) such that the following estimate holds for all pairs (x, y) such that either x or y is in E :*

$$k^{-2n} |K_k(x, y)|^2_{k\phi+\phi_{F,t}} \leq Ce^{-\sqrt{k}d(x,y)/C}$$

for all k, where $d(x, y)$ is the distance function with respect to a fixed smooth metric ω on X. Moreover, given a fixed smooth weight ϕ_{F_0} on F and a constant C' the constant C may be taken to be independent on the weight ϕ_F on F as long as $\|\partial^\alpha(\phi_F - \phi_{F_0})\|_\infty \leq C'$ for all multiindices α of total degree at most two.

Proof. Fix a point x in X and take an element s_k in $\mathcal{H}_k$ such that

$$|s_k|^2 e^{-k\phi} = |K_k(x, \cdot)|^2 e^{-k\phi(x)} e^{-k\phi(\cdot)} \tag{5.19}$$

Next, fix a point y in the set E appearing in the formulation of the theorem and "normal" local coordinates z centered at y and a "normal" trivialization of L (see the beginning of the section). In particular, $\phi(0) = \overline{\partial}\phi(0) = 0$. Identify s_k with a local holomorphic function in the $z-$variable. By the mean value property of holomorphic functions

$$s_k(0) = \int \chi_k s_k,$$

where $\chi_k = c_n k^n \chi(\sqrt{k}z)$ has unit mass and is expressed in terms of a radial smooth function χ supported on the unit-ball (so that χ_k is supported on the scaled unit ball of radius $1/\sqrt{k}$). Writing $\chi_{k\phi} := \chi_k e^{k\phi(x)}$ the relation 5.19 gives,

$$|s_k|_{k\phi}(y) = \left| \langle \chi_{k\phi}, s_k \rangle_{k\phi} \right| = |\Pi_k(\chi_{k\phi})(x)|_{k\phi}(x)$$

using the definition of s_k in the last equality. Decomposing $\Pi_k(\chi_{k\phi}) = \chi_{k\phi} + (\Pi_k(\chi_{k\phi}) - \chi_{k\phi})$ and applying Theorem 4.2 combined with proposition 4.4 now yields the following estimate

$$|s_k|_{k\phi+\sqrt{k}\psi_k}(y) \leq |\chi_{k\phi}|_{k\phi+\sqrt{k}\psi_k}(x) + Ck^{(n-1)/2} \left\|\overline{\partial}\chi_{k\phi}\right\|_{k\phi+\sqrt{k}\psi_k} \tag{5.20}$$

for any function ψ_k satisfying the assumptions in Theorem 4.2. The idea now is take ψ_k to be comparable to the distance to x. In the following we will denote by R a sufficiently large (but fixed constant).

Case 1: $d(x, y) \geq 1/R$. Set $\psi_k = \psi$ for a fixed smooth function ψ on X such that $\psi(\cdot) = 1/R$ when $d(x, \cdot) \geq 1/(2R)$ and $\psi(\cdot) = 0$ for when $d(x, \cdot) \leq 1/(4R)$. For $R >> 1$ (but fixed) the assumptions on ψ_k in

Theorem 4.2 are clearly satisfied (using that y is in the pseudo-interior of the bulk). Hence, the estimate 5.20 gives

$$|s_k|^2_{k\phi}\, e^{\sqrt{k}/C}(y) \leq 0 + Ck^n \frac{1}{k} \left\|\overline{\partial}\chi_{k\phi}\right\|^2_{k\phi+0} \leq C'k^{2n}$$

using that $\psi = 0$ on the support of $\chi_{k\phi}$ and that $\left|k\overline{\partial}\phi\right|^2$ is uniformly bounded there (since $\overline{\partial}\phi$ is assumed to be Lipschitz continuous and vanishing when $z = 0$). Since by definition s_k is related to K_k by the relation 5.19 this proves the theorem in this case.

Case 2: $d(x, y) \leq 1/R$. In this case we may assume that x is contained in the fixed coordinate neighborhood of y. By a translation of the coordinates z we now assume that they are centered at x. Set

$$\psi_k(z) = \frac{1}{R}\kappa(|z|^2 + 1/k)^{1/2}$$

where κ corresponds to a smooth function on X which is equal to one on the "ball" $\{d(,y) \leq 2/C\}$ and is supported in the set E. Accepting for for the moment that the assumptions on ψ_k in Theorem 4.2 are satisfied, the inequality 5.20 gives (with $z \leftrightarrow y$)

$$|s_k|^2_{k\phi}\, e^{\sqrt{k}(|z|^2+1/k)^{1/2})}(z) \leq |\chi_{k\phi}|^2_{k\phi+1}(x) + C'k^n \frac{1}{k} \left\|\overline{\partial}\chi_{k\phi}\right\|^2_{k\phi+1} \leq C''k^{2n}$$

using that $\sqrt{k}\psi_k \geq \sqrt{k}/\sqrt{k}$ on the support of $\chi_{k\phi}$ in the first inequality. In particular,

$$|s_k|^2_{k\phi}(z) \leq C'k^{2n}e^{-\sqrt{k}|z|}$$

which proves the theorem, since the distance function $d(\cdot, y)$ is comparable, close to y, with the distance function induced by the local Euclidean metric.

Finally, let us check that the assumptions on ψ_k in Theorem 4.2 are indeed satisfied. Differentiating gives

$$\overline{\partial}\psi_k = \frac{1}{R}(\overline{\partial}\kappa \cdot (|z|^2 + 1/k)^{1/2} - \kappa\frac{zd\bar{z}}{2(|z|^2 + 1/k)^{1/2}}) \tag{5.21}$$

Hence,

$$\left|\overline{\partial}\psi_k\right| \leq \frac{1}{R}(C' + C''\sqrt{k}...) \tag{5.22}$$

so that (i) in Theorem 4.2 holds for $R >> 1$. Next, note that $f_k := (|z|^2 + 1/k)^{1/2}$ is a psh function. Hence, formula 5.21 combined with Leibniz rule gives

$$\partial\overline{\partial}\psi_k \geq \partial\overline{\partial}\kappa \cdot f_k + \partial\kappa \wedge \overline{\partial}f_k + \overline{\partial}\kappa \wedge \partial f_k$$

and 5.22 (which clearly also holds when ψ_k is replaced by f_k) then shows that assumption (ii) in Theorem 4.2 holds, as well (even without taking R large). $\square$

5.4. **Fluctuations.**

Theorem 5.8. *Let L be a big line bundle and K_k the Bergman kernel of $\mathcal{H}(X, kL+F)$. Let u be a $\mathcal{C}^1-$function supported in the interior of the bulk. Then*

$$\frac{1}{2}\int_{X\times X} k^{-(n-1)} |K_k(x,y)|^2_{k\phi+\phi_F} (u(x)-u(y))^2 \to \|du\|^2_{dd^c\phi}$$

when k tends to infinity. Moreover, if ϕ_F satisfies the assumptions in the previous theorem, then the left hand side above is uniformly bounded by a constant independent of ϕ_F.

Proof. Denote by E the support of u which by assumption is in the bulk. First note that the integrand vanishes if both x and y are in $X-E$. We rewrite the integral above as follows:

$$2I_k := \int_{E\times X\cup X\times E} \left|k^{1/2}(u(y)-u(x))\right|^2 k^{-n} |K_k(x,y)|^2 e^{-k\phi(x)}e^{-k\phi(x)}\omega_n(x)\wedge\omega_n(y),$$

Decompose the integral above as $A_{k,R}+B_{k,R}+C_{k,R}$ according to the following three regions:

First region $(1 \leq d(x,y))$*:* By symmetry we may assume that $x \in E$. But then **??** shows that A_k tends to zero only using that u is bounded.

Second region $(Rk^{-1/2} \leq d(x,y) \leq 1)$: Again, by symmetry we may assume that $x \in E$. Since u is Lipschitz continuous $|u(y)-u(x)| \leq Cd(x,y)$. Hence, by Theorem 5.7

$$B_{k,R} \leq C\int_{Rk^{-1/2}\leq\{d(x,y)\leq 1} \left|\sqrt{k}d(x,y)\right|^2 k^n e^{-\sqrt{k}d(x,y)/C}\omega_n(x)\wedge\omega_n(y).$$

Performing a change of variables (with y fixed) then gives

$$I_k \leq C\int_X (\int_{2\sqrt{k}\geq|\zeta|\geq R/2} |\zeta|^2 e^{-|\zeta|}d\zeta ...)\omega_n(x) \to 0, \tag{5.23}$$

when first k and then R tends to infinity. □

Third region $(d(x,y) \leq Rk^{-1/2})$: Upon removing a set of measure zero we may assume that x is in the bulk (since E is a compact set in the pseudo-interior of the bulk). Take "normal coordinates" z and trivializations of L and F centered at x. Then the integral over $\{x\}\times Y$ may be written as

$$\int_{|z|\leq R} \left|k^{1/2}(u(k^{-1/2}z)-u(0))\right|^2 k^{-2n} |K_k(0,z)|^2 e^{-k\phi(z)}e^{-k\phi(z)}\omega_n(0) \tag{5.24}$$

where we have also performed the change of variables $z \to k^{-1/2}z$ in the integral. By the scaling asymptotics **??** this integral in turn equals

$$\det\lambda\int_{|z|\leq R} \left|\frac{\partial}{\partial z_i}u(0)z_i + k^{-1/2}O(|z|^2)\right|^2 (\frac{\det\lambda}{\pi^n})^2 e^{-\langle\lambda z,z\rangle}dz... + o(1).$$

Observe that

$$(\frac{\det\lambda}{\pi^n})\int_{|z|\leq R}\left|\frac{\partial}{\partial z_i}u(0)z_i+k^{-1/2}O(|z|^2)\right|^2(\frac{\det\lambda}{\pi^n})e^{-\langle\lambda z,z\rangle}dz...\to$$

$$(\frac{\det\lambda}{\pi^n})\sum_i 2\left|\frac{\partial}{\partial z_i}u(0)\right|^2\lambda_i^{-2}c_n,$$

when first k and then R tend to infinity, where

$$c_n=(\int_0^\infty se^{-s}ds)^n=-\frac{d}{dt}\mid_{t=1}\int_0^\infty e^{-ts}ds=1$$

Hence, since by lemma 5.5 the integrand in 5.24 is uniformly bounded (on $X\times X$) the dominated convergence theorem now gives

$$\lim_{R\to\infty}\lim_{k\to\infty}C_{k,R}=\int_X|du|^2_{(dd^c\phi)}(dd^c\phi)^n/n!$$

The total contribution: Finally, let first k tend to infinity (while the parameter R is kept fixed) and then R tend to infinity. Then we get

$$\lim_{k\to\infty}I_k:=\lim_{R\to\infty}\lim_{k\to\infty}(A_{k,R}+B_{k,R}+C_{k,R})=0+0+$$

$$+\int_X|du|^2_{(dd^c\phi)})(dd^c\phi)^n$$

To prove the last statement of the theorem consider the total contribution from the second and third region, i.e. the set where e $d(x,y)\leq 1$. By the uniformity obtained from the last statement in Theorem 5.7 the following bound on this contribution holds:

$$B_{k,R}+C_{k,R}\leq C\int_{0\leq\{d(x,y)\leq 1}\left|\sqrt{k}d(x,y)\right|^2k^ne^{-\sqrt{k}d(x,y)/C}\omega_n(x)\wedge\omega_n(y).$$

But a slight modification of the argument used above in the second region shows that the latter integral is uniformly bounded. This finishes the proof of the theorem.

6. Asymptotics for linear statistics

Let us first recall the setup in section 5. A line bundle $L\to X$ and a pair (ϕ,ω_n) induces a Hilbert space $\mathcal{H}(X,L)$ of dimension N with associated Bergman kernel $K(x,y)$. Recall also that, in general, a subindex k on an object indicates that it is defined with respect to $(kL,k\phi)$. Hence, we will set $k=1$ in the following definitions.

We define the associated ensemble (X^N,γ) by letting γ be the probability measure with the following density:

$$\mathcal{P}(x_1,...,x_N):=\frac{1}{N!}\det(K(x_i,x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))}).$$

By lemma 5.1 this is indeed a well-defined probability measure. Note that the ensemble is symmetric in the sense that $\mathcal{P}(x_1,...,x_N)$ is invariant under permutations of the components x_i.

6.1. **Correlation functions.** Next, we recall a general formalism of correlation functions. But it should be pointed out that in the present paper we will mainly consider the correlation functions in formula 6.1 below, that the reader could also take as definitions.

For a general symmetric ensemble (X^N, γ) the *$m-$point correlation measures* on X^m may be defined as $N!/(N-m)!$ times the pushforward of γ to X^m under the projection $(x_1, ...x_N) \mapsto (x_1, ..., x_m)$. The *$m-$point correlation functions* $\rho^{(m)}$ on X^m are then defined as the corresponding densities. As is well-known [22, 60] the fact that the kernel K represents an orthogonal projection operators leads to the following quite remarkable identities in the present context:

$$\rho^{(m)}(x_1, ..., x_m) = \det_{1\leq i,j\leq m}(K(x_i, x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))})$$

A crucial role in the present paper is played by the so called *connected $2-$point correlation function* $\rho^{(2),c}$ which may be defined by

$$\rho^{(2).c}(x, y) := \rho^{(2)}(x, y) - \rho^{(1)}(x)\rho^{(1)}(y)$$

Hence, $\rho^{(1)}$ and $\rho^{(2).c}$ may be simply expressed as

$$\rho^{(1)}(x) = |K(x, x)|_{\phi}, \quad \rho^{(2).c}(x, y) = -|K(x, y)|^2_{\phi}. \tag{6.1}$$

Remark 6.1. The present setup is essentially a special case of the general formalism of determinantal random point processes [60, 38, 40]. It falls into the class of such processes where the correlation kernel is the integral kernel of an orthogonal projection operator.

6.2. **Linear statistics.** A given (measurable) function u on (X, ω_n) induces the following random variable $\mathcal{N}[u]$ on $(X^N, d\mathcal{P})$:

$$\mathcal{N}[u](x_1, ..., x_N) := u(x_1) + ... + u(x_N).$$

Hence, if u is the characteristic function of a set Ω in X, then $\mathcal{N}[u](x_1, ..., x_n)$ simply counts the number of x_i contained in Ω. However, we will mainly consider the situation when u is $\mathcal{C}^1-$smooth. For a given random variable $\mathcal{X}$ we will write its *fluctuation* as the random variable

$$\widetilde{\mathcal{X}} := \mathcal{X} - \mathbb{E}(\mathcal{X}),$$

so that $\mathbb{E}(\widetilde{\mathcal{X}}) = 0$. Recall that the variance of a random variable $\mathcal{X}$ is defined as

$$\mathrm{Var}(\mathcal{X}) := \mathbb{E}((\widetilde{\mathcal{X}})^2)$$

The following essentially well-known lemma relates the Laplace transform of the law of $\mathcal{N}[u]$ to a Gram-Determinant.

Lemma 6.2. *Let (s_i) be an orthonormal base for $H^0(X, L)$ w.r.t. (ϕ, ω_n). Then*

$$\mathbb{E}(e^{-t\mathcal{N}[u]}) = \det\left(\langle s_i, s_j\rangle_{\phi+tu}\right)_{i,j}$$

Proof. When $u = 0$ this is just a reformulation of the integral formula in lemma 5.1. The general case then follows by expressing the base (s_i) in terms of a ON-base w.r.t $(\phi + tu, \omega_n)$ and using a simple transformation property of the corresponding determinants. $\square$

The following lemma is also essentially well-known. For a proof in the present context ones can use the previous lemma and results in the first reference in[13].

Lemma 6.3. *The following formulas for the expectation and variance of $\mathcal{N}_k[u]$ hold:*

$$(i)\ \mathbb{E}_{\phi+tu}(\mathcal{N}[u]) = -\frac{d}{dt}\log \mathbb{E}_{\phi+tu}(e^{-t\mathcal{N}[u]}) = \int_X |K_{\phi+tu}(x,x)|_{\phi+tu}\, u(x)$$

and

$$(ii)\ \ Var_{\phi+tu}(\mathcal{N}[u])) = \frac{d^2}{d^2t}\log \mathbb{E}_{\phi+tu}(e^{-t\mathcal{N}[u]}) =$$

$$= \frac{1}{2}\int_{X\times X} |K_{\phi+tu}(x,y)|^2_{\phi+tu}\,(u(x)-u(y))^2\omega_n(x)\wedge\omega_n(y)$$

Proposition 6.4. *Suppose that u is a bounded function on X and (ϕ,μ) is a general weighted measure. Then*

$$(i)\ \ Var_k(\mathcal{N}[u])) = O(k^n)$$

Moreover, if (ϕ,ω_n) is strongly regular and u continuous, then

$$(ii)\ \ Var_k(\mathcal{N}[u])) = o(k^n).$$

Proof. By (ii) in lemma 6.3

$$\mathrm{Var}_k(\mathcal{N}[u])) = \frac{1}{2}\int_{X\times X} |K_k(x,y)|^2_{k\phi}\,(u(x)-u(y))^2\omega_n(x)\wedge\omega_n(y)$$

The first item of the proposition follows immediately, since u is assumed bounded, from combining 5.6 and 5.4 and using that $N_k = O(k^n)$ for *any* line bundle L. The second item follows from [9] where it is shown that

$$\int k^{-n}\,|K_k(x,y)|^2_{k\phi}\, f(x)g(y)\omega_n(x)\wedge\omega_n(y) \to \int_X fg\mu_{\phi_e},$$

for any continuous functions f, g. $\square$

6.3. **A law of large numbers (proof of Thm 1.4).** By (i) in Lemma 6.3 and one of the main results in [13]:

$$\mathbb{E}_k(k^{-n}\mathcal{N}[u]) = \int_X |K_k|_{k\phi}\, u\omega_n \to \int_X u\mu_{\phi_e}.$$

Moreover, by (i) in the previous proposition

$$\mathrm{Var}_k(k^{-n}\mathcal{N}[u])) = o(k^{-n}) \to 0.$$

Hence, the theorem follows directly from Chebishevs inequality, just like in the usual proof of the classical weak law of large numbers.

6.4. **A central limit theorem (proof of Thm 1.5).**

Proof. Let $\mathcal{F}_k(t) := -\log \mathbb{E}_k(e^{-tk^{-(n-1)/2}\widetilde{\mathcal{N}_k}[u]})$. By (i) in Lemma 6.3

$$\frac{d\mathcal{F}_k(t)}{dt}_{t=0} = k^{-(n-1)/2}\mathbb{E}_k(\widetilde{\mathcal{N}_k}) = 0, \tag{6.2}$$

using the definition of $\widetilde{\mathcal{N}_k}$ in the last equality. Moreover, by (ii) in Lemma 6.3

$$\frac{d^2\mathcal{F}_k(t)}{d^2t} = -k^{-(n-1)}\frac{1}{2}\int_{X\times X} |K_{k\phi+th_k}(x,y)|^2_{k\phi+th_k}(h_k(x)-h_k(y))^2$$

where $h_k = u - c_k$ with $c_k = \mathbb{E}_k(\mathcal{N}_k)$. Next, note that the map $\psi \mapsto |K_\psi(x,y)|^2_\psi$ is clearly invariant under $\psi \to \psi + c$ for any constant c. Hence, we get

$$\frac{d^2\mathcal{F}_k(t)}{d^2t} = -\frac{1}{2}\int_{X\times X} |K_{k\phi+tu}(x,y)|^2_{k\phi+tu}(u(x)-u(y))^2$$

Applying Theorem 5.8 to $kL+F$ where F is the trivial holomorphic line bundle equipped with the weight $k^{-(n-1)/2}tu$ (taking for example $\phi_{F_0}\equiv 0$) gives

$$\lim_{k\to\infty}\frac{d^2\mathcal{F}_k(t)}{d^2t} = -\left\|du\right\|^2_{dd^c\phi}$$

for all t. Using that $\mathcal{F}_k(t)$ is concave with a uniform bound on the second derivatives (by the uniformity in Theorem 5.8) and 6.2 the theorem now follows by integrating over t and using the dominated convergence theorem. □

6.4.1. *Proof of the corollary.* Item (i) follows immediately from the preceding theorem by evaluating the convergence of the second derivatives at $t=0$ and using lemma 6.3. To prove item (ii) set $\nu_k := k^{-(n-1)/2}\widetilde{\mathcal{N}_k}[u]_*(\gamma_k)$, which gives a sequence of probability measures on $\mathbb{R}$, obtained by pushing forward the probability measure γ_k. In the notation of the previous theorem:

$$f_k(t) = \left\langle \nu_k, e^{-t(.)}\right\rangle$$

which gives a well-defined holomorphic function for all t in $\mathbb{C}$ with

$$|f_k(t)| \leq C_K\left\langle \nu_k, 1\right\rangle = C$$

for all $t\in\mathbb{C}$ such that $|t|\leq R$. But by the proof of the preceding theorem $f_k(t)\to f(t)$, where $f(t)$ is an entire function, on the maximally totally real set $\mathbb{R}$ in $\mathbb{C}$. Hence, the same normal families argument as below formula 5.18 shows that uniform convergence actually holds on compacts of $\mathbb{C}$ (even for all derivatives). Setting $t=i\xi$ with $\xi\in\mathbb{R}$ in particular gives that the Fourier transforms $\widehat{\nu_k}$ converges uniformly om compacts in $\mathbb{R}_\xi$ towards $\widehat{\nu}$, where $\widehat{\nu}$ (and hence ν) is a centered Gaussian. But as is well-known this latter fact is equivalent to the stated convergence in distribution.

7. Free energy and phase transitions

Fix a weakly regular weighted measure (ϕ_0, ω_n) and write S_k for a fixed ordered base in the corresponding Hilbert space $H^0(X, kL)$. Recall from section 5 that this gives rise to an element $\det(S_k)$ in $H^0(X^{N_k}, (kL)^{\boxtimes N_k})$. The *free energy* of the weight $k\phi$ (relative $k\phi_0$) is now defined as

$$\mathcal{F}[k\phi] := -\log(\|\det S_k\|^2_{k\phi})$$

Note that if $\phi = \phi_0 + u$ Lemma 6.2 gives that

$$\mathcal{F}[k\phi] := -\log \mathbb{E}_{k\phi_0}(e^{-k\mathcal{N}[u]}) \tag{7.1}$$

Fix a $\mathcal{C}^{1,1}-$ family of weight ϕ_t as in section 3.1, where t is in the unit-disc and write $\mathcal{F}_k(t) := \mathcal{F}[k\phi_t]$.

7.1. **Convergence of first derivatives (proof of Theorem 1.1.3).** A simple modification of (i) in lemma 6.3 with $u := \frac{\partial \phi_t}{\partial t}_{t=0}$ gives

$$k^{-(n+1)}\frac{\partial \mathcal{F}_k(t)}{\partial t}_{t=0} = k^{-n}\mathbb{E}_{k\phi_0}(\mathcal{N}[u]) \to \int_X \mu_{\phi_0,e} u,$$

where we have used a consequence of Theorem 1.4 in the last step. But by formula 3.9 the right hand side is precisely equal to $\frac{\partial \mathcal{F}_\infty[\phi_t]}{\partial t}_{t=0}$ and since by definition $\mathcal{F}_k(0) = \mathcal{F}_\infty(0) = 0$ this finishes the proof of the theorem.

7.2. **Convergence of second derivatives (proof of Theorem 1.7).** Combining Theorem 5.8 and a simple modification of Lemma 6.3 now gives with $u := \frac{\partial \phi_t}{\partial t}_{t=0}$

$$\lim_{k\to\infty} k^{-(n+1)}\frac{\partial^2 \mathcal{F}_k(t)}{\partial t \partial \bar{t}}_{t=0} = \lim_{k\to\infty} k^{-n}\mathbb{E}_{k\phi_0}(\frac{\partial^2 \phi_t}{\partial t \partial \bar{t}}_{t=0}) - \|du\|^2_{dd^c\phi} = \tag{7.2}$$

where the first term by Theorem 1.4 equals $\int_X \frac{\partial^2 \phi_t}{\partial t \partial \bar{t}}_{t=0}(dd^c\phi)^n/n!$. But by proposition 3.6 the right hand side in 7.2 is then precisely $\frac{\partial^2 \mathcal{F}_\infty(t)}{\partial t \partial \bar{t}}_{t=0}$, which finishes the proof.

Remark 7.1. The convergence of the free energy, expressed in formula 7.1, may also be also formulated in terms of the theory of large deviations (see [35] and references therein) as follows. Denote by $\mathcal{P}(X)$ the space of all probability measures on X. Then the *law* of the normalized random measure in formula 1.6 is a probability measure ν_N on $\mathcal{P}(X)$ (i.e. the pushforward of the probability measure γ_N on X^N under the normalized map defined by formula 1.6). The convergence of the free energy in Theorem 1.1.3 may hence be formulated, using the formalism in remark 3.7, as

(7.3)

$$-k^{-(n+1)}\log \int_{\mathcal{P}(X)} e^{k^{n+1}U[\nu_N]}\nu_N \to \inf{}_\mu(I'_{\phi-u}[\mu] = \inf{}_\mu(I'_\phi[\mu] - U[\mu]),$$

where the infimum is taken over all measures μ in the subset $\mathcal{E}_1(X)$ of $\mathcal{P}(X)$. Now, if we knew that ν_N satisfied a *large deviation property with*

rate functional $-I'_\phi$ *(see* [35] *for the definition)* then 7.3 would follow from Varadhan's lemma which applies to *all* bounded continuous functions U on $\mathcal{P}(X)$ (and not only to linear ones). In general, Bric's theorem gives the reverse statement. Hence, to obtain the large deviation property one would have to extend the convergence 7.3 to other functions U on $\mathcal{P}(X)$ than linear ones. It should also be pointed out that one usually demands that the rate functional be semi-continous and defined on *all* of $\mathcal{P}(X)$. Anyway, it seems natural to conjecture that ν_N has a large deviation property with (some extension of) the rate functional $-I'_\phi$.

7.3. **Variation of weights and direct image bundles.** In this section we will take $k = 1$ and consider the line bundle $L + K_X$ over X. Recall (section 3.1) that the family ϕ_t of weights on $L \to X$ may be identified with a $\mathcal{C}^{1,1}-$smooth weight Φ on the pulled-back line bundle $\pi_X^* L$ on $\Delta \times X$ under the natural projection from $\Delta \times X$ to X. Let $\mathcal{F}(t) := \mathcal{F}[\phi_t]$, which according to Lemma 6.2 may be written in terms of a Gram-Schmidt matrix. Following [14] we may hence write

$$\frac{\partial^2 \mathcal{F}(t)}{\partial t \partial \bar{t}} = \mathrm{Tr}\Theta_t,$$

where Θ_t is an N times N Hermitian matrix which may be identified with the curvature matrix of the vector bundle E obtained as a direct image bundle. Concretely, this means that E is the Hermitian holomorphic vector bundle over Δ, whose fiber E_t over t is the space $H^0(X, L + K_X)$ equipped with the inner product induced by the weight ϕ_t.[3]

The following proposition is a generalization to non-psh weights of Berndtsson's curvature estimate for direct image bundles in [14].

Proposition 7.2. *Let ϕ_t be as above and such that $\frac{\partial}{\partial t}\phi_t$ is for each fixed t supported in the pseudo-interior of the bulk of (X, ϕ_0). Then*

$$\left\langle \Theta^{H^0} s, s \right\rangle_{\phi_t} \geq \langle c(\phi_t) s, s \rangle_{\phi_t}$$

for any element s in $H^0(X, L + K_X)$, where

$$c(\phi_t) = t_*(dd^c \Phi)^{n+1} / (d_X d_X^c \phi_t)^n dt \wedge d\bar{t}(n+1)!$$

Proof. The proof can be given word by word as in [14] , if one replaces the use of the Hörmander-Kodaira estimate there with the estimate in Theorem 4.1. The details will appear elsewhere. □

In particular, the previous proposition implies that the free energy $\mathcal{F}(t)$ is spsh as a function of t under the conditions on ϕ_t in the previous theorem combined with the condition that $\phi_t(z)$ be locally spsh w.r.t. (t, z) when z is in the bulk of (X, ϕ_0) (ensuring that $c(\phi_t)$ is positive).

[3] Hence, $\mathrm{Tr}\Theta_t$ is the curvature form of the corresponding determinant line bundle whose fiber over t is the top exterior power $\bigwedge^N H^0(X, L + K_X)$.

It is also interesting to see that a similar argument gives the general estimate

$$\langle \Theta s, s \rangle_{\phi_t} \leq 0 \tag{7.4}$$

for *any* continuous affine family $\phi_t = \phi + tu$ if $E_t = H^0(X, L)$ (we have fixed a reference measure μ). Geometrically, this just amounts to the fact that curvature of holomorphic bundles decreases along subbundles. In particular, 7.4 implies the well-known fact that $\mathcal{F}(\phi + tu)$ is concave w.r.t. a real parameter t (which is also a consequence of the fact that the second derivatives are given by minus the corresponding variance).

Remark 7.3. Note that summing the inequality in proposition 7.2 over an orthonormal base (s_i) gives with L replaced by kL

$$k^{-(n+1)} \frac{\partial^2 \mathcal{F}_k(t)}{\partial t \partial \bar{t}} = \mathrm{Tr}\Theta_t \geq \int_X c(\phi_t) k^{-n} \rho_k^{(1)} \omega_n \to t_* (dd^c \Phi)^{n+1}/(n+1)!,$$

where we have used formula 1.4 in the last step. This inequality is consistent, as it must, with Theorem 1.7. It should be pointed out that in the case that ϕ is smooth and spsh the asymptotic equality in Theorem 1.7 follows from Theorem 4.2 in [14] (see also [46]), where the subleading term in the expansion is also computed. Yet another proof, which bypasses the decay estimate in Theorem 1.3 could be obtained by using the formula in Lemma 6.3(ii) as before and estimating the l.h.s. above from below by integrating merely over the "third region" in $X \times X$ defined in the proof of Theorem 5.8 and then proceeding as before. This then gives the corresponding upper bound on the l.h.s. which hence must be an asymptotic equality.

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